Cocycles, radicals and splitting fields of twisted group algebras

Hans Opolka
TU Braunschweig
Universitätsplatz 2
D-38106 Braunschweig
e-mail: h.opolka@tu-bs.de

Abstract: For a field \( k \) of characteristic 0, a finite group \( G \) and a central 2-cocycle \( f: G \times G \to k^* \) denote by \( (k, G, f) \) the corresponding twisted group algebra. The purpose of this note is to show that a certain radical extension of \( k \), which is constructed from \( f \), is a splitting field of \( (k, G, f) \), and to illustrate this result by an example which is related to Gauss sums.

Key words: Semisimple algebras, splitting fields

MSC classification 2010: 16K20, 16S35, 20C25

For basic definitions and results about group cohomology and twisted group algebras resp. about general finite dimensional associative algebras which will be used in this note see e.g. [Y], [CO] resp. [BR], [CR]. For the case of the ordinary group algebra \( f = 1 \) we also refer to [SE] and [LA], XVIII.

Let \( k \) be a field of characteristic 0, let \( C \) be an algebraic closure of \( k \), let \( G \) be a finite group, let \( f: G \times G \to k^* \) be a central 2-cocycle and denote by \( (k, G, f) \) the corresponding twisted group algebra. As is well known it is semisimple. Consider the function \( a_f: G \to k^* \) defined by

\[ a_f(x) := \prod_{i=1}^{m(x)} f(x, x^i), \quad x \in G, \]

where \( m(x) \) denotes the order of an element \( x \in G \). For every \( x \in G \) fix the set of all roots \( \alpha_f(x) \in C \) of \( a_f(x) \) of order dividing \( m(x) \) and denote by \( L/k \) the subextension of \( C/k \) which is obtained from \( k \) by adjoining to \( k \) all \( \alpha_f(x), x \in G \). The main result of this note is as follows

(1) Theorem The field \( L \) is a splitting field of \( (k, G, f) \)

We note that this result, applied to \( k = \mathbb{Q} \) and \( f = 1 \), contains the well known result of R. Brauer, see e.g. [SE], 12.3 or [LA], XVIII, §11, Thm. 17, according to which the field extension of \( \mathbb{Q} \) which is obtained from \( \mathbb{Q} \) by adjoining to \( \mathbb{Q} \) a primitive \( m \)-th root of unity
\[ \zeta \in \mathbb{C}, \text{ where } m \text{ is the exponent of } G, \text{ is a splitting field of the ordinary group algebra } \mathbb{Q}[G] = (\mathbb{Q}, G, 1). \]

In the proof of (1) we will use some results on representations of twisted group algebras. The first is the twisted version of the reciprocity law (1.12) in [O1] in terms of characters:

Let \( f, f' : G \times G \to k^* \) be central 2-cocycles, let \( H \leq G \) be a subgroup, let \( \chi \) be a \( k - f \) -character of \( G \), i.e. the character of a representation of \((k, G, f)\), viewed as a function \( \chi : G \to k \), and let \( \gamma \) be a \( k - f' \) -character of \( H \). Then

\[ (2) \quad \chi \cdot \text{Ind}_{H}^{G}(\gamma)|_H = \text{Ind}_{H}^{G}((\chi|_H \cdot \gamma)|_H)_{f,f'} \cdot \]

Here \( \text{Ind}_{H}^{G}(\theta)|_H \) denotes the \( k - t \) - character of \( G \) which is induced by the \( k - t \) - character \( \theta \) of \( H \) with respect to the central 2-cocycle \( t : G \times G \to k^* \), i.e.

\[ \text{Ind}_{H}^{G}(\theta)|_H(x) = \frac{1}{|H|} \cdot \sum_{g \in G} \frac{t(g,x)t(gx,g^{-1})}{t(g,g^{-1})} \theta^*(gxg^{-1}), \quad x \in G, \]

where \( \theta^*(z) = \theta(z) \) for \( z \in H \) and \( \theta^*(z) = 0 \) for \( z \in G \setminus H \).

Denote by \( W_k \) the group of roots of unity in \( k \). It is well known, see e.g. [O1], (1.2) and (1.4), that there is a subgroup \( F \) of \( k^* \) and an isomorphism

\[ H^2(G, k^*) \cong H^2_{\text{abel}}(G/G', F) \times H^2(G, W_k); \]

here \( G' \) denotes the commutator subgroup of \( G \), cohomology is taken with respect to the trivial group action and \( H^2_{\text{abel}}(G/G', F) \) denotes the subgroup of cocycle classes which can be represented by symmetric cocycles. Hence, up to equivalence of cocycles, we may and do assume that there is a central symmetric cocycle \( s' : G/G' \times G/G' \to F \) and a central cocycle \( t : G \times G \to W_k \) such that \( f = s \cdot t \) where \( s = \text{inf}_{G}^{G}(s') \). Now Schur’s lemma implies there is a function \( \alpha : G \to C^* \) such that \( s = \delta \alpha \cdot t \) (coboundary over \( C \)) and therefore \( f = \delta \alpha \cdot t \). It follows that for every (simple) \( C - f \) - character \( \chi : G \to C \) there is a (simple) \( C - t \) - character \( \psi : G \to C \) such that \( \chi = \alpha \cdot \psi \) and such that \( \psi \) belongs to a \( C - t \) - representation of \( G \) which can be lifted to a linear representation \( D \) of a finite central group extension \( 1 \to Z \to E \to G \to 1 \) which is defined by \( t \). If \( D \) is induced by a representation \( D_0 \) of a subgroup \( H_0 \leq E \) such that \( Z \leq H_0 \), then \( \psi \) is induced by a \( C - t \) - character \( \psi_0 \) of the subgroup \( H := H_0/Z \leq G \) with respect to \( t \), and the degree of \( \psi_0 \) is the degree of \( D_0 \). If \( G \) is nilpotent, then \( E \) is nilpotent. Assume that \( \chi \) is simple. Then \( \psi \) is simple. Since finite nilpotent groups are monomial there is a subgroup \( H \leq G \) and a function \( \beta : H \to C^* \) such that \( \delta \beta = t|_{H} \cdot H \) and \( \psi = \text{Ind}_{H}^{G}(\beta)|_H \). Hence from (2) we obtain

\[ \chi = \alpha \cdot \psi = \text{Ind}_{H}^{G}((\alpha|_H \cdot \beta)|_H). \]

This proves the following lemma.
(3) Lemma If $G$ is nilpotent then every simple $C - f -$ character $\chi$ of $G$ is monomial, i.e. there is a subgroup $H \leq G$ and also a function $h : H \to C^*$ such that $\delta h = f|_{H \times H}$ and $\chi = Ind_H^G(h)_f$.

We note

$$y(x)^{m(x)} = \prod_{i=1}^{m(x)} f(x, x^i) = a_f(x)$$

for all $x \in H$ where $m(x)$ is the order of $x$. Hence $Ind_H^G(h)_f$ is the $C - f -$character of a representation of $G$ which is realizable in the subfield $L = k(\sqrt[1]{a_f(x)} : x \in G)$ of $C$. Using Brauer’s induction theorem [BT] or [SE], §10, Thm. 19 or [LA], XVIII, § 10, Thm. 15, and the twisted form (2) of the reciprocity law, a twisted form of Brauer’s induction theorem was obtained in [O1], p. 584, and combining this result with (3) we obtain the following proposition.

(4) Proposition For every $C - f -$ character $\chi$ of $G$ there are nilpotent subgroups $H_1, \ldots, H_r$ of $G$ and functions $\gamma_1 : H_1 \to C^*, \ldots, \gamma_r : H_r \to C^*$ such that the coboundary $\delta \gamma_i$ is the restriction of $f$ to $H_i \times H_i$ for all $i = 1, \ldots, r$ and such that there are integers $n_1, \ldots, n_r$ with the property

$$\chi = \sum_{i=1}^r n_i Ind_{H_i}^G(\gamma_i)_f$$

We add that every $Ind_{H_i}^G(\gamma_i)_f$ is the character of a $C - f -$representation of $G$ which is realizable over the field $L$. In order to complete the proof of (1) one argues as in the proof of the linear case $f = 1$, comp. e.g. [LA], XVIII, § 10, proof of Thm. 17: Decompose every $Ind_{H_i}^G(\gamma_i)_f$ in (4) as a sum of simple characters of $C - f -$representations of $G$ which are realizable over $L$ to obtain an expression of $\chi$ as a linear combination with nonnegative integer coefficients of simple characters which belong to simple representations of $(C, G, f)$ which are realizable over $L$. This shows that $\chi$ itself belongs to a representation of $(C, G, f)$ which is realizable over $L$ and therefore completes the proof of (1).

Finally we discuss an example. The basic construction is taken from [O2], § 4; it makes use of relations between central 2-cocycles and bimultiplicative pairings which are explained in [Y], §2, and of elementary facts about Gauss sums. Let $m$ be a positive integer $> 1$ and let $W_m = \langle e^{2\pi i/m} \rangle \leq C^*$ be the group of roots of unity of order $m$ in $C$. Assume that $A$ is a finite abelian group of exponent $m$ and that $t : A \times A \to W_m$ is a central 2-cocycle such that the associated symplectic pairing $\omega_t : A \times A \to W_m$, $\omega_t(x, y) := t(x, y)/(t(y, x)$ for all $x, y \in A$, which is defined e.g. in [Y], §2, 2.1, (7), is nondegenerate. Le $h$ denote a positive integer such that there is an epimorphism $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q}) \to A$. For every character $\chi$ of
A, viewed as a character of $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q})$, denote by $\tau(\chi)$ the corresponding Gauss sum, i.e.

$$\tau(\chi) := \sum_{a \mod f(\chi)} \chi(a)e^{2\pi i a / f(\chi)}$$

where $f(\chi)$ is the conductor of $\chi$; for the terminology and elementary results on Gauss sums which are used here compare [LE], § 2. For every $x \in A$ let $\chi_x$ denote the character of $A$ defined by $\chi_x(y) := \omega_t(x,y)$, $y \in A$, viewed as a character of $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q})$. Put $k := \mathbb{Q}(e^{2\pi i/m})$. Then $f: A \times A \to k^*$ defined by

$$f(x,y) := t(x,y)\tau(\chi_x)\tau(\chi_y)/\tau(\chi_x\chi_y), \ x,y \in A,$$

is a central 2-cocycle. Now we assume that the central 2-cocycle $t: A \times A \to \mu_m$ is a bimultiplicative pairing; see [Y], § 2, Thm. 2.2. Then the function $a_f: A \to k^*$ defined above is given by

$$a_f(x) = t(x,x)^{\frac{m(x)(m(x)+1)}{2}} \tau(\chi_x)^{m(x)}, \ x \in A.$$

We have $\varepsilon(x) := t(x,x)^{\frac{m(x)(m(x)+1)}{2}} \in \{\pm 1\}$ for all $x \in A$, and $\alpha_f(x) = \frac{m(x)}{\sqrt{\varepsilon(x)}} \cdot \tau(\chi_x), \ x \in A$, is a root in $\mathbb{C}$ of order dividing $m(x)$ of $a_f(x)$. Hence the splitting field $L$ of the twisted group algebra $(k, A, f)$ from (1) is given by

$$L = k(\sqrt[\varepsilon(x)]{\sqrt{\varepsilon(x)}} \cdot \tau(\chi_x): x \in A).$$

Denote by $l$ the lcm of $h$ and $m$. Since $\tau(\chi_x) \in \mathbb{Q}(e^{2\pi i/l})$ for every $x \in A$ we have $L \subset \mathbb{Q}(e^{\pi i/l})$. Especially $\mathbb{Q}(e^{\pi i/l})$ is a splitting field of the twisted group algebra $(k, A, f)$.

References


[O2] H. Opolka: Cocycles, Galois theory and automorphic forms, parshin70.mi.ras.ru/materials.html
