Some results connected to Dedekind's Zeta functions

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Some results connected to Dedekind's Zeta functions

By Robert W. van der Waall, Amsterdam

On behalf of the 150th anniversary of R. Dedekind's birthday on 6 October 1831 we present three results in which Dedekind zeta functions and Artin L-functions are involved. In a) we recall the main contents of our paper [4]. In b) the construction of a holomorphic Artin L-function $L(s,\chi, K/k)$ is given for which $\chi$ is a non-monomial irreducible character of a certain group $G$. In c) we regard the group $GL(2,3)$ and we construct some relations among Artin L-functions derived from it.


a) (1977, [F], pages 649–662; see also [4] in which the explicit proofs are given)

Let $L$ be an algebraic number field (= extension field of finite degree of the rational field $\mathbb{Q}$). Let

$$\zeta_L(s) = \sum_a (Na)^{-s},$$

defined in the domain $\text{Re}(s)>1$, the summation being extended over all integral ideals of $L$. The function $\zeta_L(s)$, $s$ complex variable, is called the zeta function of Dedekind. The zeta function $\zeta_L(s)$ converges absolutely and uniformly in the domain $\text{Re}(s) \geq 1+\delta$, any $\delta > 0$. It was E. Hecke, who found the functional equation for $\zeta_L(s)$ and he proved that $\zeta_L(s)$ has an analytic continuation over the whole complex plane, with the sole exception at the point $s=1$, where there is a simple pole. Notice that $\zeta_{\mathbb{Q}}(s)$ is just the ordinary Riemann zeta function.

In [1], E. Artin poses the following conjecture.

CONJECTURE. Let $M$ be a finite extension of $L$. Then $\zeta_M(s)/\zeta_L(s)$ is holomorphic in the whole complex plane.

In my 1977-survey paper in [F] I gave some cases for which the conjecture becomes a theorem. The main result was the following

THEOREM. Let $\Omega \supseteq M \supseteq L$ be fields with $[\Omega : L]<\infty$. Suppose that $\Omega/L$ is a galois extension with $\text{Gal}(\Omega/L)$ a solvable group. Then $\zeta_M(s)/\zeta_L(s)$ is a holomorphic function.

The proof is based on the following proposition proved in [4].
PROPOSITION. Let \( X \triangleright Y \triangleright L \) be fields with \( [X : L] < \infty \) and assume that \( X/L \) is a galois extension with galois group \( G = \text{Gal}(X/L) \). Furthermore assume that \( H = \text{Gal}(X/Y) \) is a maximal subgroup of \( G \) and that \( G = HA \), with \( A \) some normal abelian subgroup of \( G \) with \( H \cap A = \{1\} \). Let \( \psi \) be the principal character of \( H \). Then

\[
\psi^G = e + \sum_{i=1}^{t} \varepsilon_i,
\]

where \( e \) is the principal character of \( G \) and where the \( \varepsilon_i \) are non-principal monomial irreducible characters of \( G \); here \( t \geq 1 \) and \( \varepsilon_j \neq \varepsilon_m \) if \( j \neq m \). Therefore, by Artin's formalism,

\[
\zeta_Y(s)/\zeta_L(s) = \prod_{i=1}^{t} L(s, \varepsilon_i, X/L)
\]

is a holomorphic function.

b) Let \( r \) be a prime number with \( r \equiv 3 \pmod{4} \). Let \( E \) be an extra special \( r \)-group of order \( r^2 \) and of exponent \( r \), that is,

\[
E = \langle s, t, z \mid s^r = t^r = z^r = [s, z] = [t, z] = 1, z = [s, t] \rangle.
\]

The quaternion group \( Q \) of order 8 acts on \( E \) as follows. Let

\[
Q = \langle p, q \mid p^4 = 1, p^2 = q^2, q^{-1}pq = p^{-1} \rangle.
\]

It is well known that every element of any finite field is a sum of two squares in that field. Thus we can choose

\[
\alpha, \beta \in \mathbb{Z} \text{ with } \alpha^2 + \beta^2 = -1 \pmod{r}.
\]

Let \( G \) be the relative holomorph of \( E \) with \( Q \). The concept is clear. It follows that

\[
x^p = x, \text{ whence } Z(E) = Z(G).
\]

Let \( \chi \in \text{Irr } E \) be an irreducible character of \( E \) with \( \chi(1) = r \). Set \( s^p = s^{\alpha t^\beta}, t^p = s^{\beta t^{-\alpha}}, z^p = t, q^p = s^{-1}. \)

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Let \( \chi \in \text{Irr } E \) be an irreducible character of \( E \) with \( \chi(1) = r \). Set \( s^p = s^{\alpha t^\beta}, t^p = s^{\beta t^{-\alpha}}, s^p = t, t^q = s^{-1}. \)
irreducible character $\eta \in \text{Irr } E < p^2 >$ for any $i = 1, 2, 3, 4$ with $\eta = \chi$. Hence $\chi < p^2 > = \eta + (\eta \otimes \delta)$, where $\eta \otimes \delta \in \text{Irr } E < p^2 >$, $\eta \neq \eta \otimes \delta$, $\delta(f) = 1$ for all $f \in E$, $\delta(f) = -1$ for all $f \in E < p^2 > - E$. Let $\varphi \in \text{Irr } G$ have $\eta \otimes \delta$ in its restriction to $E < p^2 >$. Hence $\chi$ is contained in $\varphi_E$. Therefore $\varphi \in \{ \chi \otimes \psi_i | i = 1, 2, 3, 4, 5 \}$ and we see that $\varphi = \chi \otimes \psi_5$.

It follows that

$$((\chi \otimes \psi_5)_{E < p^2 >}, \eta) = (\chi \otimes \psi_5, \eta^G) = (\chi \otimes \psi_5, \sum_{i=1}^{4} (\chi \otimes \psi_i)) = 0.$$  

Thus $(\chi \otimes \psi_5)_{E < p^2 >} = 2(\eta \otimes \delta)$. By Frobenius' reciprocity for groups we have now $(\eta \otimes \delta)^G = 2(\chi \otimes \psi_5) + \ldots$, but also $(\eta \otimes \delta)^G = 4(\eta \otimes \delta)(1) = 2((\chi \otimes \psi_5)(1))$. Hence $(\eta \otimes \delta)^G = 2(\chi \otimes \psi_5)$. Now $\mu < p^2 > = \chi < p^2 > = \eta + (\eta \otimes \delta) = (\mu^H) < p^2 > = \xi_1 < p^2 > + \xi_2 < p^2 >$. As $\eta \in \text{Irr } E < p^2 >$ and $\eta \otimes \delta \in \text{Irr } E < p^2 >$ we can choose $\xi_2 < p^2 > = \eta \otimes \delta$. Hence $\xi_2^G = (\eta \otimes \delta)^G = 2(\chi \otimes \psi_5)$. Let $G$ be the galois group of the galois extension $M/L$. Let $\Omega$ be the invariant field for $H$. Then $L(s, \xi_2, M/\Omega) = L(s, \xi_2^G, M/L) = L(s, 2(\chi \otimes \psi_5), M/L) = (L(s, \chi \otimes \psi_5, M/L))^2$ is a holomorphic function as $L(s, \xi_2, M/\Omega)$ is a Hecke–Dirichlet L-function because of $\xi_2^G(1) = 1$. A result of R. Brauer yields that $L(s, \chi \otimes \psi_5, M/L)$ is a meromorphic function, see Theorem V.19.3 of [H]. Hence $L(s, \chi \otimes \psi_5, M/L)$ itself is now holomorphic. We have $(\chi \otimes \psi_5)(1) = 2r$. We shall show that $G$ does not contain subgroups of index $2r$, that is, $\chi \otimes \psi_5$ is not a monomial character. Indeed, let $U \subset G$ be a subgroup of $G$ with index $2r$. Since $E$ is a normal subgroup of $G$, we have $EU/E = U/(E \cap U)$. If $G = UE$, then $2r = |G : U| = |UE : U| = |E : (E \cap U)|$ divides $|E| = r^3$, a contradiction. Therefore $G \neq UE$, i.e. $UE : U| = |E : (E \cap U)|$ divides $|E| = r^3$. Since $E$ is the Sylow $r$-subgroup of $G$, $UE \neq U$. Thus $|G : UE| = 2$. Now $UE$ is equal to $E < p >$, $E < q >$ or $E < pq >$ and this exhaust all possibilities. However, $< p >$, $< q >$ and $< pq >$ operate irreducibly by conjugation on $E/Z(E)$, as $p^2 = q^2 = (pq)^2$ inverts any element of $E/Z(E)$ and as $r \equiv 3 \pmod{4}$. Therefore such a group $U$ does not exist.

Finally we note that the case $r = 3$ was found by J. G. Thompson (letter to J. P. Serre, dated July 27, 1974).

c) We refer to [3]. Let $G = \text{GL}(2, 3)$. The group $G$ is solvable and $|G| = 48$. According to I. Schur (1907) $G$ has the following character table.

<table>
<thead>
<tr>
<th>representant of</th>
<th>number of el.</th>
<th>$\psi_1$</th>
<th>$\psi_2$</th>
<th>$\psi_3$</th>
<th>$\psi_4$</th>
<th>$\psi_5$</th>
<th>$\psi_6$</th>
<th>$\psi_7$</th>
<th>$\psi_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{(10)}$ $01$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$^{(-10)}$ $01$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>$^{(0-1)}$ $10$</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$^{(01)}$ $-11$</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>
Let $b = (\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix})$. Hence $|\langle b \rangle| = 6$. Let $\alpha \in \text{Irr} \langle b \rangle$ with $\alpha(1) = \alpha(b^2) = \alpha(b^4) = 1$, $\alpha(b) = \alpha(b^3) = \alpha(b^5) = -1$. Notice that $b^2 = (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})$. It follows that $\alpha^G = 2\psi_8$. Let $e = e^{2\pi i/6}$. Define $\gamma(b^t) = e^t$ for $1 \leq t \leq 6$ and write $\bar{\gamma}(b^t) = \gamma(b^t)$; whence $\gamma, \bar{\gamma} \in \text{Irr} \langle b \rangle$.

We have

$$\gamma^G = \psi_3 + \psi_4 + \psi_8 = \bar{\gamma}^G.$$  

Therefore $2(\psi_3 + \psi_4) = \gamma^G + \bar{\gamma}^G - \alpha^G$.

Next, let $M/S$ be a galois extension with $G \cong \text{Gal}(M/S)$. Then it follows that

$$(L(s, \psi_3 + \psi_4, M/S))^2 = \frac{L(s, \gamma + \bar{\gamma}, M/U)}{L(s, \alpha, M/U)},$$

where $U$ is the invariant field of the group $\langle b \rangle$, considered as subgroup of $\text{Gal}(M/S)$. The $L$-functions $L(s, \alpha, M/U)$ and $L(s, \gamma + \bar{\gamma}, M/U)$ are holomorphic, following Hecke. So not only for $\Re(s) \geq 1$, but also for any $s \in \mathbb{C}$ $(L(s, \psi_3 + \psi_4, M/S))^2$ is defined by the last formula. We know by a result of R. Brauer that $L(s, \psi_3 + \psi_4, M/S)$ is meromorphic, and it follows also that $L(s, \psi_8, M/S)$ is holomorphic, as $\psi_8$ is monomial. We drop the symbols $M, S, U$ for the moment.

Now observe that $\alpha(b^3) = \gamma(b^3) = \bar{\gamma}(b^3) = -1$. Hence the functional equation for Artin $L$-functions as given in [2], page 306, gives the following formulae.

$$L(1-s, \gamma) = W(\gamma) \left( N^\mathbb{F} (\gamma, M/U) \right)^{-\frac{1}{2}} A(s, \gamma) L(s, \gamma) \quad \text{(x)}$$

$$L(1-s, \alpha) = W(\alpha) \left( N^\mathbb{F} (\alpha, M/U) \right)^{-\frac{1}{2}} A(s, \alpha) L(s, \alpha) \quad \text{(xx)}$$

Just as $\alpha(b^3) = \gamma(b^3)$, it follows that $A(s, \alpha) = A(s, \gamma)$. The reader is referred to the explicit formulae for $A(s, \gamma)$ (and for $A(s, \alpha)$) in [2]. We have $\alpha = \bar{\alpha}$ and $L(s, \gamma) = L(s, \bar{\gamma})$ by $\gamma^G = \bar{\gamma}^G$. Keep in mind that both $L(s, \alpha)$ and $L(s, \gamma)$ are zero-point-free in $\Re(s) \geq 1$.

It follows that

$$L(s, \alpha) A(s, \alpha) W(\alpha) \left( N^\mathbb{F} (\alpha, M/U) \right)^{-\frac{1}{2}} L(1-s, \gamma) =$$

$$= W(\gamma) \left( N^\mathbb{F} (\gamma, M/U) \right)^{-\frac{1}{2}} A(s, \gamma) L(s, \bar{\gamma}) L(1-s, \alpha).$$

As $A(s, \alpha) = \text{product of cos-, sin-, and } \Gamma$-functions, we see that

$$L(s, \alpha) W(\alpha) \left( N^\mathbb{F} (\alpha, M/U) \right)^{-\frac{1}{2}} L(1-s, \gamma) =$$

$$= W(\gamma) \left( N^\mathbb{F} (\gamma, M/U) \right)^{-\frac{1}{2}} L(s, \bar{\gamma}) L(1-s, \alpha) \quad \text{xx)}$$
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for those s for which $A(s, \alpha) \neq 0$. However, if $A(s_0, \alpha) = 0$, then $L(1-s_0, \alpha) = 0$ as $L(s, \alpha)$ is analytic for all $s \in \mathbb{C}$, following formula $x)$. The same holds for formula $x)$. Hence formula $\frac{1}{2}$ holds for all $s \in \mathbb{C}$. Notice that $L(s, \gamma) = L(s, \gamma) = L(s, \psi_3 + \psi_4) L(s, \psi_9)$ for all $s \in \mathbb{C}$. Further $L(s, \alpha) = (L(s, \psi_8))^2$ for any $s \in \mathbb{C}$. Hence we find that

$$L(s, \psi_8) L(1-s, \psi_8) \left\{ L(s, \psi_8) W(\alpha) \left( N \delta (\alpha, M/U) \right)^{s-\frac{1}{2}} L(1-s, \psi_3 + \psi_4) - W(\gamma) \left( N \delta (\gamma, M/U) \right)^{s-\frac{1}{2}} L(s, \psi_3 + \psi_4) L(1-s, \psi_8) \right\} = 0.$$ 

Write in short $L(s, \psi_8) L(1-s, \psi_8) A(s) = 0$. Now, if $L(s_1, \psi_8) = 0$ for $\frac{1}{2} \leq \text{Re}(s_1) < 1$, then also $L(1-s_1, \psi_8) = 0$ by the functional equation. Hence $A(s_1) = 0$. For $|\text{Re}(s_1)| \geq 1$ we can only have $L(1-s_1, \psi_8) = 0$ by the functional equation. Hence $A(s_1) = 0$. Look at $x)$. Then $L(1-s_1, \gamma) = 0$ and by $x)$ again, the order of the zero $s_i$ in $L(1-s_1, \gamma)$ is equal to the order of the zero $s_i$ in $L(1-s_1, \alpha)$. Therefore

$$\left( L(1-s_i, \psi_3 + \psi_4) \right)^2 = \frac{L(1-s_i, \gamma))^2}{L(1-s_i, \alpha)} = 0.$$

The final result is that $A(s) = 0$ for all $s \in \mathbb{C}$, as now $L(1-s_1, \psi_3 + \psi_4) = 0$.

Next we look at the several Artin root numbers involved here. By the corollary on page 18 of [F] we have

$$W(\gamma) = W(\psi_3 + \psi_4 + \psi_8) = W(\psi_3) W(\psi_4) W(\psi_8) \text{ and } W(\alpha) = W(2 \psi_8) = (W(\psi_8))^2.$$

The Artin root numbers are complex roots of unity and it holds that $W(\psi_4)$ is the complex conjugate to $W(\psi_3)$ as $\psi_4 = \overline{\psi_3}$. Hence $W(\psi_3) W(\psi_4) = 1$ and so $W(\gamma) = W(\psi_8)$. The character $\psi_8$ is afforded by a real representation of $GL(2, 3)$. Thus we can apply a theorem of A. Fröhlich and J. Querut, see [F] page 124, that says that in such a situation $W(\psi_8) = 1$. Hence $W(\gamma) = W(\psi_8) = 1 = (W(\psi_8))^2 = W(\alpha)$.

Therefore we have proved that

$$L(s, \psi_8) L(1-s, \psi_3 + \psi_4) \left( N \delta (\alpha, M/U) \right)^{s-\frac{1}{2}} = \frac{L(1-s, \psi_8) L(s, \psi_3 + \psi_4) \left( N \delta (\gamma, M/U) \right)^{s-\frac{1}{2}}}{L(1-s, \psi_8) L(s, \psi_3 + \psi_4) \left( N \delta (\gamma, M/U) \right)^{s-\frac{1}{2}}}.$$

References