A new approach to the real numbers (motivated by continued fractions)

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A new approach to the real numbers (motivated by continued fractions)

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Introduction

There are several methods known of extending the ordered field $\mathbb{Q}$ of the rational numbers to the complete ordered field $\mathbb{R}$ of the real numbers. In this paper we give a new and very natural method for this extension; the motivation comes from the theory of continued fractions. We define the set $\mathbb{R}\setminus\mathbb{Q}$ of the irrational numbers as the set of all infinite sequences $<a_0, a_1, a_2, \ldots>$ with $a_0 \in \mathbb{Z}$, $0 < a_j \in \mathbb{Z}$ $(j > 0)$. By this the set $\mathbb{R} := \mathbb{Q} \cup (\mathbb{R}\setminus\mathbb{Q})$ is given in an explicit and simple form at the very beginning and we believe that this approach is an important advantage over all other extensions of $\mathbb{Q}$ to $\mathbb{R}$. After this we study ordering, completeness, and arithmetical operations for the set $\mathbb{R}$. It is clear that all methods of extending $\mathbb{Q}$ to $\mathbb{R}$ have some common features since the result, namely $\mathbb{R}$ and its structure, is always the same.

In § 1 we bring known facts concerning the continued fraction expansion of rational numbers. In § 2 we introduce $\mathbb{R}$ by our method as an ordered set which we call K for caution's sake and we prove the theorem of the supremum for K. Afterwards K can be made a commutative additive group with $\mathbb{Q}$ as subgroup in § 3, a division ring with $\mathbb{Q}$ as subring in § 4, and finally a field with $\mathbb{Q}$ as subfield in § 5; there addition, subtraction, multiplication, and division, as far as they go beyong $\mathbb{Q}$, are defined by using the supremum. Finally, we write $\mathbb{R}$ instead of K.

§ 1. Rational numbers and finite continued fractions

Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$; the fraction $\frac{a}{b}$ is called reduced if and only if $(a,b) = 1$. Every rational number can be written in exactly one way as a reduced fraction.

A finite sequence $<a_0, a_1, \ldots a_n>$ with

$$n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ a_0 \in \mathbb{Z}, \ a_j \in \mathbb{N} \ (0 < j \leqq n)$$

is called a finite chain. The set of all finite chains we denote by E. A finite chain is called normed, if and only if in case $n > 0$ we have $a_n > 1$. The set of all normed finite chains we denote by E'. We have $E' \subset E$. We define the map

$$\Phi: E \to \mathbb{Q}$$

by

$$\Phi(<a_0, a_1, \ldots, a_n>) := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}; \qquad (1.1)$$

the right hand side of this equation is called finite continued fraction. Let $\frac{a}{b} \in \mathbb{Q}$; suppose the euclidean algorithm for a, b takes the form

$$\begin{array}{ll} a = ba_0 + r_1, & 0 < r_1 < b, \\ b = r_1 a_1 + r_2, & 0 < r_2 < r_1, \\ r_1 = r_2 a_2 + r_3, & 0 < r_3 < r_2, \\ \vdots & \\ r_{n-2} = r_{n-1} a_{n-1} + r_n & 0 < r_n < r_{n-1}, \\ r_{n-1} = r_n a_n + 0; & \end{array}$$

we obtain a map

$$\Delta\colon \mathbb{Q} \to E'$$

by

$$\Delta\left(\frac{a}{b}\right) := \langle a_0, a_1, \ldots, a_n \rangle.$$

Elimination in the euclidean algorithm gives

$$\frac{a}{b} = \Phi(\langle a_0, a_1, \ldots, a_n \rangle).$$

Consequently, we have

$$\mathbb{Q} \xrightarrow{\Delta} E' \xrightarrow{\Phi} \mathbb{Q} = \mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q}, \qquad E' \xrightarrow{\Phi} \mathbb{Q} \xrightarrow{\Delta} E' = E' \xrightarrow{\mathrm{id}} E'.$$

Especially, the restriction of Φ to E' is bijective. Since

$$\Phi(\langle a_0, \ldots, a_{n-2}, a_{n-1}, 1 \rangle) = \Phi(\langle a_0, \ldots, a_{n-2}, a_{n-1} + 1 \rangle) \quad (n > 0),$$

Φ itself is not injective. We are here mainly interested in $\mathbb{Q}$; with respect to $\mathbb{Q}$ we do not lose anything by

Convention 1. Any finite chain $\langle a_0, \ldots, a_{n-2}, a_{n-1}, 1 \rangle$ with $n > 0$ has to be replaced by $\langle a_0, \ldots, a_{n-2}, a_{n-1} + 1 \rangle \in E'$. Furthermore, we identify $\langle a_0, \ldots, a_n \rangle \in E'$ and $\Phi(\langle a_0, \ldots, a_n \rangle) \in \mathbb{Q}$.

For $\alpha = \langle a_0, \ldots, a_n \rangle \in \mathbb{Q}$, $j \in \mathbb{N}_0$ let

$$(1.2) \qquad \alpha^{(j)} := \begin{cases} \langle a_0, a_1, \ldots, a_j \rangle & \text{in case } j < n \\ \alpha & \text{in case } j \geqq n; \end{cases}$$

let furthermore

$$\left.\begin{array}{l} p_0 := 0,\ p_1 := 1,\ p_j := a_j p_{j-1} + p_{j-2} \\ q_0 := 1,\ q_1 := a_1,\ q_j := a_j q_{j-1} + q_{j-2} \end{array}\right. \quad (1 < j \leqq n),$$
$$p_j := p_n,\ q_j := q_n \ (j > n).$$

We have

$$\alpha^{(j)} = a_0 + \frac{p_j}{q_j} \qquad (j \geqq 0),$$

$$p_{j-1}q_j - p_jq_{j-1} = (-1)^j \quad (0<j\leqq n),$$
$$p_jq_{j-2} - p_{j-2}q_j = (-1)^j\, a_j \quad (1<j\leqq n),$$

(1.3) $\alpha^{(0)} \leqq \alpha^{(2)} \leqq \alpha^{(4)} \leqq \ldots \leqq \alpha \leqq \ldots \leqq \alpha^{(5)} \leqq \alpha^{(3)} \leqq \alpha^{(1)} \leqq \alpha^{(0)} + 1$

(with = up to at most $n+1$ exceptions),

$$\alpha^{(j+1)} - \alpha^{(j)} = \tfrac{(-1)^j}{q_jq_{j+1}} \quad (0\leqq j<n).$$

Following Fibonacci let

$$F_0 := 1,\ F_1 := 1,\ F_j := F_{j-1} + F_{j-2} \quad (j>1).$$

Induction gives

$$F_jF_{j+1} \geqq 2^j \quad (j\geqq 0);$$

by $q_j \geqq F_j$ $(j\geqq 0)$ we conclude

(1.4) $|\alpha^{(j+1)} - \alpha^{(j)}| \leqq 2^{-j} \quad (j\geqq 0).$

$\alpha = <a_0,\ldots,a_n> \in \mathbb{Q}$ and $\beta = <b_0,\ldots,b_m> \in \mathbb{Q}$ can easily be compared in size. In order to avoid case distinctions in case $n \neq m$ we introduce the symbol ω with the property $r<\omega$ or equivalently $\omega>r$ $(r\in\mathbb{Q})$.

Convention 2. For every $\alpha = <a_0,\ldots,a_n> \in \mathbb{Q}$ let $a_j := \omega$ $(j>n)$ and hence $\alpha = <a_0,\ldots,a_n, \omega, \omega, \ldots>$.

Obviously we have

Lemma 1.1. Let

$$\alpha = <a_0,\ldots,a_n, \omega, \omega,\ldots> \in \mathbb{Q},$$
$$\beta = <b_0,\ldots,b_m, \omega, \omega,\ldots> \in \mathbb{Q},$$

$\alpha \neq \beta$; we define $k = k(\alpha,\beta) \in \mathbb{N}_0$ by

$$a_j = b_j \ (0\leqq j<k),\ a_k \neq b_k;$$

then we have

(1.5) $\alpha<\beta \Leftrightarrow \begin{cases} a_k<b_k \text{ in case } 2\,|\,k \\ a_k>b_k \text{ in case } 2\nmid k. \end{cases}$

Here we have $k(\alpha,\beta) = k(\beta,\alpha) \leqq \sup\{n,m\}$.

§ 2. The ordered set K

We extend the set $\mathbb{Q}$ to the set K by adjoining as new elements all infinite sequences $<a_0,a_1,a_2,\ldots>$ with $a_0 \in \mathbb{Z}$, $a_j \in \mathbb{N}$ $(j>0)$.

For $<a_0,a_1,a_2,\ldots> \in K$, $m\in\mathbb{N}$ we have

(2.1) $a_m = \omega \Rightarrow a_j = \omega \ (j>m).$

Let $\alpha = <a_0,a_1,a_2,\ldots> \in K$, $\beta = <b_0,b_1,b_2,\ldots> \in K$, $\alpha \neq \beta$. We extend the definition of $k(\alpha,\beta)$ of Lemma 1.1 to $\alpha \notin \mathbb{Q} \vee \beta \notin \mathbb{Q}$. We have $k(\alpha,\beta) = k(\beta,\alpha)$. For $\alpha \notin \mathbb{Q} \vee \beta \notin \mathbb{Q}$ we use (1.5) as **Definition 2.1** of $\alpha < \beta$ or equivalently of $\beta > \alpha$.

For $\alpha \in K$, $\beta \in K$ we have

(2.2) $\alpha < \beta \vee \alpha = \beta \vee \alpha > \beta$, exclusively.

Furthermore, let $\gamma \in K$; then we have

(2.3) $(\alpha < \beta \wedge \beta < \gamma) \Rightarrow \alpha < \gamma$ (transitivity of $<$).

Let $\alpha = \langle a_0, a_1, a_2, \ldots \rangle \in K \backslash \mathbb{Q}$, $j \in \mathbb{N}_0$; we extend (1.2) and let

$$a^{(j)} := \langle a_0, a_1, \ldots, \alpha_j \rangle,$$

where we observe Convention 1 and possibly Convention 2; instead of (1.3) we have

(2.4) $\alpha^{(0)} < \alpha^{(2)} < \alpha^{(4)} < \ldots < \alpha < \ldots < \alpha^{(5)} < \alpha^{(3)} < \alpha^{(1)} \leqq \alpha^{(0)} + 1.$

We have $\omega \notin K$ since $\omega = \alpha \in K$ gives the contradiction $\omega \leqq \alpha^{(0)} + 1 \in \mathbb{Z}$.

Let $\alpha \in K$, $\beta \in K$, $\alpha \neq \beta$, $k := k(\alpha,\beta)$. (1.5) implies

$$\alpha < \beta \Rightarrow (\alpha^{(j)} = \beta^{(j)}\ (0 \leqq j < k) \wedge \alpha^{(j)} < \beta^{(j)}\ (j \geqq k)).$$

By (2.2) this implies

$$\left(\underset{i \in \mathbb{N}_0}{\exists} \alpha^{(i)} < \beta^{(i)} \right) \Rightarrow \alpha < \beta.$$

We need the consequences

(2.5) $\begin{cases} \alpha^{(2j)} \leqq \beta^{(2j)}\ (j \geqq 0) \Leftrightarrow \alpha \leqq \beta \Leftrightarrow \alpha^{(2j+1)} \leqq \beta^{(2j+1)}\ (j \geqq 0), \\ 0 \leqq \beta \Leftrightarrow 0 \leqq \beta^{(0)},\ 0 < \beta \Leftrightarrow 0 < \beta^{(2)}. \end{cases}$

Let $M \subset K$, $M \neq \emptyset$; $\tau \in K$ is called upper bound of M if and only if $\alpha \leqq \tau$ ($\alpha \in M$); M is called bounded above if and only if there exists at least one upper bound of M; an upper bound σ of M is called supremum (or least upper bound) of M if and only if every upper bound τ of M satisfies $\sigma \leqq \tau$. M has at most one supremum.

Theorem 2.1 of the supremum. Every $M \subset K$, $M \neq \emptyset$, which is bounded above, has exactly one supremum in K and we denote it by $\sup M$.

Proof. We construct $\sigma = \sup M$. For $M \cap \mathbb{Q}$ we observe Convention 2. We use repeatedly the well-ordering of $\mathbb{Z}$. Let $\emptyset \neq A \subset \mathbb{N}$; denote by $v(A)$ the minimal element of A; in case A is bounded above, denote by $w(A)$ the maximal element of A; in case A is not bounded above, let $w(A) := \omega$; let also

$$v(A \cup \{\omega\}) := v(A),\quad v(\{\omega\}) := \omega,$$
$$w(A \cup \{\omega\}) := \omega,\quad w(\{\omega\}) := \omega.$$

For $\alpha = \langle a_0, a_1, a_2, \ldots \rangle \in K$ we have

$$a_0 \leqq \langle a_0, a_1, a_2, \ldots \rangle < a_0 + 1.$$

$M^{(0)} := M$ is bounded above and so is

$$M^{[0]} := \{a_0 \colon \alpha \in M\} \subset \mathbb{Z};$$

we have $M^{[0]} \neq \emptyset$; denote by s_0 the maximal element of $M^{[0]}$. Let

$$M^{(1)} := \{\alpha \in M^{(0)} \colon a_0 = s_0\}.$$

We have $\emptyset \neq M^{(1)} \subset M^{(0)}$,

$$M^{[1]} := \{a_1 \colon \alpha \in M^{(1)}\} \neq 0, \; s_1 := v(M^{[1]}).$$

In case $s_1 = \omega$ we are done and put

$$\sigma := \langle s_0, \omega, \omega, \ldots \rangle.$$

In case $s_1 \neq \omega$ we go on and let

$$M^{(2)} := \{\alpha \in M^{(1)} \colon a_1 = s_1\}.$$

We have $\emptyset \neq M^{(2)} \subset M^{(1)}$,

$$M^{[2]} := \{a_2 \colon \alpha \in M^{(2)}\} \neq 0, \; s_2 := w(M^{[2]}).$$

In case $s_2 = \omega$ we are done and put

$$\sigma := \begin{cases} \langle s_0, s_1, \omega, \omega, \ldots \rangle & \text{in case } s_1 > 1 \\ \langle s_0 + 1, \omega, \omega, \omega, \ldots \rangle & \text{in case } s_1 = 1. \end{cases}$$

In case $s_2 \neq \omega$ we go on and let

$$M^{(3)} := \{\alpha \in M^{(2)} \colon a_2 = s_2\}.$$

We have $\emptyset \neq M^{(3)} \subset M^{(2)}$,

$$M^{[3]} := \{a_3 \colon \alpha \in M^{(3)}\} \neq 0, \; s_3 := v(M^{[3]}).$$

In case $s_3 = \omega$ we are done and put

$$\sigma := \begin{cases} \langle s_0, s_1, s_2, \omega, \omega, \ldots \rangle & \text{in case } s_2 > 1 \\ s_0, s_1 + 1, \omega, \omega, \omega, \ldots \rangle & \text{in case } s_2 = 1. \end{cases}$$

In case $s_3 \neq \omega$ we go on and let

$$M^{(4)} := \{\alpha \in M^{(3)} \colon a_3 = s_3\}.$$

We have $\emptyset \neq M^{(4)} \subset M^{(3)}$,

$$M^{[4]} := \{a_4 \colon \alpha \in M^{(4)}\} \neq 0, \; s_4 := w(M^{[4]}).$$

In case $s_4 = \omega$ we are done and put

$$\sigma := \begin{cases} \langle s_0, s_1, s_2, s_3, \omega, \omega, \ldots \rangle & \text{in case } s_3 > 1 \\ \langle s_0, s_1, s_2 + 1, \omega, \omega, \omega, \ldots \rangle & \text{in case } s_3 = 1. \end{cases}$$

In case $s_4 \neq \omega$ we go on. In this fashion we have defined

$$\sigma = \langle s_0, s_1, s_2, \ldots \rangle \in K$$

by a terminating or non-terminating construction where w( ) and v( ) have been used alternately.

Let $\alpha \in M$, $\alpha \neq \sigma$. For $k := k(\alpha,\sigma)$ (as after (2.1)) we have

(2.6) $a_j = s_j\ (0 \leqq j < k),\ a_k \neq s_k$;

in the construction above $M^{(k)}$ appears by (2.1) and we have $\alpha \in M^{(k)}$; by definition of s_k we have

(2.7) $\begin{cases} a_k < s_k & \text{in case } 2 \mid k \\ a_k > s_k & \text{in case } 2 \nmid k; \end{cases}$

hence $\alpha < \sigma$, and σ is an upper bound of M.

Let $\alpha \in K$, $\alpha < \sigma$; by $\alpha < \sigma$ we have (2.6) and (2.7); since $a_k \neq \omega$ in case $2|k$ and since $s_k \neq \omega$ in case $2 \nmid k$ it follows $s_j \neq \omega$ $(0 \leqq j < k)$ by (2.1), and in the construction above certainly

$$\begin{cases} M^{(1)} & \text{in case } k = 0 \\ M^{(k+1)} & \text{in case } 2 \nmid k \\ M^{(k)} & \text{in case } 2 \mid k \wedge k > 0 \end{cases}$$

appears.

Case $k = 0$. Every $\beta \in M^{(1)}$ satisfies $\beta > \alpha$.

Case $2 \nmid k$. Every $\beta \in M^{(k+1)}$ satisfies $\beta > \alpha$.

Case $2|k \wedge k > 0$. For $M^{(k)}$ we distinguish 3 possibilities. Let firstly $s_k = \omega \in M^{[k]}$; then

$$\beta := \langle s_0, s_1, \ldots, s_{k-1}, \omega, \omega, \ldots \rangle \in M^{(k)}$$

and $\beta > \alpha$. Let secondly $s_k = \omega \notin M^{[k]}$; then there exist

$$\beta := \langle s_0, s_1, \ldots, s_{k-1}, b_k, b_{k+1}, \ldots \rangle \in M^{(k)}$$

with arbitrarily large $b_k \in \mathbb{N}$; for $b_k > a_k$ we have $\beta > \alpha$. Let thirdly $s_k < \omega$; then there exist

$$\beta := \langle s_0, s_1, \ldots, s_k, b_{k+1}, b_{k+2}, \ldots \rangle \in M^{(k)}$$

and we have $\beta > \alpha$. In every case we have found a $\beta \in M$ with $\beta > \alpha$, and hence there exists no upper bound of M which is smaller than σ.

This proves the theorem.

This proof gives beyond (2.4) also

(2.8) $\alpha = \sup\{\alpha^{(2n)} : n \geqq 0\} \qquad (\alpha \in K)$.

Theorem 2.2. For every $\alpha \in K$, $\beta \in K$ with $\alpha < \beta$ we can find $r \in \mathbb{Q}$ with $\alpha < r < \beta$.

Proof. For $\alpha \in \mathbb{Q} \wedge \beta \in \mathbb{Q}$ we take $r := \frac{\alpha+\beta}{2}$. Let $\alpha \notin \mathbb{Q} \vee \beta \notin \mathbb{Q}$, $k := k(\alpha,\beta)$, $\alpha = \langle a_0, a_1, \ldots \rangle$, $\beta = \langle b_0, b_1, \ldots \rangle$.

Case $2|k$. Then $a_k < b_k$; in case $b_{k+1} < \omega$ we choose
$r := \langle b_0, b_1, \ldots, b_k, b_{k+1}+1 \rangle$; in case $b_{k+1} = \omega$ we have $\beta \in \mathbb{Q}$, $\alpha \notin \mathbb{Q}$ and choose $r := \langle a_0, a_1, \ldots, a_{k+1}, a_{k+2}+1 \rangle$.

Case $2 \nmid k$. Then $a_k > b_k$; in case $a_{k+1} < \omega$ we choose
$r := \langle a_0, a_1, \ldots, a_k, a_{k+1}+1 \rangle$; in case $a_{k+1} = \omega$ we have $\alpha \in \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and choose $r := \langle b_0, b_1, \ldots, b_{k+1}, b_{k+2}+1 \rangle$.

§ 3. K as additive group

For $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}$ we have

(3.1) $\quad \alpha+\beta = \sup\{\alpha^{(2n)} + \beta^{(2n)}: n \geqq 0\}$.

For $\alpha \in K$, $\beta \in K$, $\alpha \notin \mathbb{Q} \vee \beta \notin \mathbb{Q}$ we use (3.1) as **Definition 3.1** of $\alpha+\beta$; here we observe

$$\alpha^{(2n)} + \beta^{(2n)} < \alpha^{(1)} + \beta^{(1)} \qquad (n \geqq 0)$$

by (1.3) and (2.4), and the Theorem of the supremum is applicable.

For $\alpha \in K$, $\beta \in K$ we have

$$\alpha + 0 = 0 + \alpha = \alpha,$$

(3.2) $\quad \alpha+\beta = \beta+\alpha \quad$ (commutativity of addition).

For α, β, γ, δ from K we have

(3.3) $\quad (\alpha \leqq \beta \wedge \gamma \leqq \delta) \Rightarrow \alpha+\gamma \leqq \beta+\delta \quad$ (monotonicity of addition);

indeed:

$$\left.\begin{array}{ll} \alpha \leqq \beta \Rightarrow \alpha^{(2n)} \leqq \beta^{(2n)} & (n \geqq 0) \text{ by } (2.5) \\ \gamma \leqq \delta \Rightarrow \gamma^{(2n)} \leqq \delta^{(2n)} & \end{array}\right\} \Rightarrow$$
$$\alpha^{(2n)} + \gamma^{(2n)} \leqq \beta^{(2n)} + \delta^{(2n)} \quad (n \geqq 0) \leqq \beta+\delta \text{ by } (3.1)$$
$$\Rightarrow \alpha+\gamma \leqq \beta+\delta \text{ by } (3.1).$$

For $\alpha \in K$, $\beta \in K$ and h, j, m, n from $\mathbb{N}_0$ we have

(3.4) $\quad \alpha^{(2h)} + \beta^{(2j)} \leqq \alpha+\beta \leqq \alpha^{(2m+1)} + \beta^{(2n+1)}$

by (1.3), (2.4), (3.3).

Theorem 3.1. For $\alpha \in K$, $\beta \in K$, $\gamma \in K$ we have

(3.5) $\quad (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma) \quad$ (associativity of addition).

Proof. We make the assumption "<"; by Theorem 2.2 there exist $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ with

$$(\alpha+\beta)+\gamma < r < s < \alpha+(\beta+\delta);$$

by (3.4) we have

$$\alpha^{(2n)} + \beta^{(2n)} \leqq \alpha+\beta,\ \gamma^{(2n)} \leqq \gamma,$$
$$\alpha \leqq \alpha^{(2n+1)},\ \beta+\gamma \leqq \beta^{(2n+1)} + \gamma^{(2n+1)} \qquad (n \geqq 0);$$

by (3.3) we obtain

$$\lambda_n := \alpha^{(2n)} + \beta^{(2n)} + \gamma^{(2n)} \leqq (\alpha+\beta) + \gamma,$$
$$\varrho_n := \alpha^{(2n+1)} + \beta^{(2n+1)} + \gamma^{(2n+1)} \geqq \alpha + (\beta+\gamma) \qquad (n \geqq 0);$$

by (2.3) we obtain in $\mathbb{Q}$ on the one hand

$$\lambda_n < r < s < \varrho_n \qquad (n \geqq 0);$$

by (1.4) we have on the other hand

$$\varrho_n - \lambda_n < 4^{1-n} \qquad (n \geqq 0);$$

for all $n \in \mathbb{N}$ with $4^{1-n} \leqq s-r$ this is a contradiction. Similarly the assumption "$>$" leads to a contradiction. Finally (2.2) gives (3.5).

For $\alpha \in K$ we have

$$-\alpha^{(2n+1)} \leqq -\alpha^{(2n+3)} \leqq -\alpha^{(0)} \qquad (n \geqq 0);$$

by (1.3) and (2.4). For $\alpha \in \mathbb{Q}$ we have

(3.6) $\quad -\alpha = \sup\{-\alpha^{(2n+1)} : n \geqq 0\}$

and $\alpha + (-\alpha) = 0$. For $\alpha \in K \setminus \mathbb{Q}$ we use (3.6) as **Definition 3.2** of $-\alpha$.

Theorem 3.2. For $\alpha \in K$ we have $\alpha + (-\alpha) = 0$.

Proof. By (3.6) we have

$$-\alpha^{(2n+1)} \leqq -\alpha \qquad (n \geqq 0).$$

By (1.3), (2.4) we have

$$-\alpha^{(2j+1)} \leqq -\alpha^{(2n)} \qquad (j \geqq 0,\ n \geqq 0)$$

and by (3.6) hence

$$-\alpha \leqq -\alpha^{(2n)} \qquad (n \geqq 0).$$

By (1.3), (2.4) we have

$$\alpha^{(2n)} \leqq \alpha \leqq \alpha^{(2n+1)} \qquad (n \geqq 0).$$

Therefore (3.3) implies

$$\alpha^{(2n)} - \alpha^{(2n+1)} \leqq \alpha + (-\alpha) \leqq \alpha^{(2n+1)} - \alpha^{(2n)} \qquad (n \geqq 0).$$

The assumption $0 < \alpha + (-\alpha) \vee \alpha + (-\alpha) < 0$ leads by Theorem 2.2, (2.3), (1.4) in $\mathbb{Q}$ to the contradiction

$$\underset{r \in \mathbb{Q},}{\exists} \ \underset{n \geqq 0}{\forall} \quad 0 < r < \alpha^{(2n+1)} - \alpha^{(2n)} \leqq 4^{-n}.$$

(2.2) gives the result.

Hence K is a commutative group with respect to + and has $\mathbb{Q}$ as subgroup.

Let $\alpha \in K$, $\beta \in K$, $\alpha - \beta := \alpha + (-\beta)$. We have

(3.7) $-(-\alpha) = \alpha,\ -(\alpha+\beta) = (-\alpha) + (-\beta) = -\alpha - \beta.$

Since

$$\begin{aligned} \alpha \leqq \beta &\Rightarrow -\beta^{(2n+1)} \leqq -\alpha^{(2n+1)} && (n \geqq 0) \text{ by } (2.5) \\ &\Rightarrow -\beta^{(2n+1)} \leqq -\alpha && (n \geqq 0) \text{ by } (3.6) \\ &\Rightarrow -\beta \leqq -\alpha && \text{by } (3.6) \end{aligned}$$

we find

(3.8) $\alpha < \beta \Leftrightarrow -\beta < -\alpha.$

§ 4. Multiplication in K

For $\alpha \in K$ we have

$$|\alpha| := \sup\{\alpha, -\alpha\} = \begin{cases} \alpha \text{ in case } \alpha \geqq 0 \\ -\alpha \text{ in case } \alpha < 0 \end{cases} \quad \text{by (3.8);}$$

we have $|\alpha| = |-\alpha| \geqq 0$. For $\alpha \in K$, $\beta \in K$ we have

$$|\alpha| = |\beta| \Leftrightarrow (\alpha = \beta \vee \alpha = -\beta).$$

For $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}$ we have

$$(4.1) \quad \alpha\beta = \begin{cases} \sup\{\alpha^{(2n)}\beta^{(2n)} : n \geqq 0\} & \text{in case } \alpha \geqq 0 \wedge \beta \geqq 0 \\ -(|\alpha| \cdot \beta) & \text{in case } \alpha < 0 \wedge \beta > 0 \\ -(\alpha \cdot |\beta|) & \text{in case } \alpha > 0 \wedge \beta < 0 \\ |\alpha| \cdot |\beta| & \text{in case } \alpha < 0 \wedge \beta < 0. \end{cases}$$

For $\alpha \in K$, $\beta \in K$, $\alpha \notin \mathbb{Q} \vee \beta \notin \mathbb{Q}$ we use (4.1) as **Definition 4.1** of $\alpha \cdot \beta$ (or shorter $\alpha\beta$); in the uppermost case we observe

$$\alpha^{(2n)}\beta^{(2n)} \leqq \alpha^{(1)}\beta^{(1)} \qquad (n \geqq 0)$$

by (1.3), (2.4) and hence the Theorem of the supremum is applicable.

Let $\alpha \in K$, $\beta \in K$; (4.1) gives

(4.2) $\alpha\beta = \beta\alpha$ (commutativity of multiplication),
$0\alpha = 0,\ 1\alpha = \alpha$ by (2.8),

$$(4.3) \begin{cases} \left.\begin{aligned} \alpha > 0 \Rightarrow \alpha^{(2)} > 0 \\ \beta > 0 \Rightarrow \beta^{(2)} > 0 \end{aligned}\right\} \Rightarrow \alpha\beta > 0 \text{ by (2.3)} \\ \alpha < 0 \wedge \beta > 0 \quad \Rightarrow \alpha\beta < 0 \text{ by (3.8)} \\ \alpha > 0 \wedge \beta < 0 \quad \Rightarrow \alpha\beta < 0 \text{ by (3.8)} \\ \alpha < 0 \wedge \beta < 0 \quad \Rightarrow \alpha\beta > 0, \end{cases}$$

$$\alpha\beta = 0 \Leftrightarrow (\alpha = 0 \vee \beta = 0) \text{ by (2.2);}$$

distinguishing 4 cases as in (4.1) we obtain

(4.4) $|\alpha\beta| = |\alpha|\,|\beta|$

(indeed: for $\alpha \geqq 0 \wedge \beta \geqq 0$ this says $\alpha\beta = \alpha\beta$; for $\alpha < 0 \wedge \beta > 0$ we have $\alpha\beta := -(|\alpha|\beta) < 0$, $|\alpha\beta| = |\alpha|\beta$ (by (3.7)) $= |\alpha|\,|\beta|$; for $\alpha > 0 \wedge \beta < 0$ we have $\alpha\beta := -(\alpha|\beta|) < 0$, $|\alpha\beta| = \alpha|\beta| = |\alpha|\,|\beta|$; for $\alpha < 0 \wedge \beta < 0$ we have $\alpha\beta := |\alpha|\,|\beta| > 0$, $|\alpha\beta| = |\alpha|\,|\beta|$;) and

(4.5) $(-\alpha)\beta = -(\alpha\beta) = \alpha(-\beta)$, $(-\alpha)(-\beta) = \alpha\beta$.

For $\alpha, \beta, \gamma, \delta$ from K we have

(4.6) $(0 \leqq \alpha \leqq \beta \wedge 0 \leqq \gamma \leqq \delta) \Rightarrow \alpha\gamma \leqq \beta\delta$ (monotonicity of multiplication);

indeed: we have $0 \leqq \alpha^{(2n)}\gamma^{(2n)} \leqq \beta^{(2n)}\delta^{(2n)}$ by (2.5) and hence $\alpha\gamma \leqq \beta\delta$.

For $\alpha \in K$, $\beta \in K$, $\alpha \geqq 0$, $\beta \geqq 0$ and h, j, m, n from $\mathbb{N}_0$ we have

(4.7) $0 \leqq \alpha^{(2h)}\beta^{(2j)} \leqq \alpha\beta \leqq \alpha^{(2m+1)}\beta^{(2n+1)}$

by (1.3), (2.4), (4.6).

Theorem 4.1. For $\alpha \in K$, $\beta \in K$, $\gamma \in K$ we have

(4.8) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associativity of multiplication).

Proof. We consider first the special case $\alpha > 0$, $\beta > 0$, $\gamma > 0$; we make the assumption "$<$" (as in the proof of (3.5)); by Theorem 2.2 there exist $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ with

$$(\alpha\beta)\gamma < r < s < \alpha(\beta\gamma);$$

by (4.7) we have

$$0 \leqq \alpha^{(2n)}\beta^{(2n)} \leqq \alpha\beta,\ 0 \leqq \gamma^{(2n)} \leqq \gamma,$$
$$0 < \alpha \leqq \alpha^{(2n+1)},\ 0 \leqq \beta\gamma \leqq \beta^{(2n+1)}\gamma^{(2n+1)} \qquad (n \geqq 0);$$

by (4.6) we obtain

$$\lambda_n := \alpha^{(2n)}\beta^{(2n)}\gamma^{(2n)} \leqq (\alpha\beta)\gamma,$$
$$\varrho_n := \alpha^{(2n+1)}\beta^{(2n+1)}\gamma^{(2n+1)} \geqq \alpha(\beta\gamma) \qquad (n \geqq 0);$$

by (2.3) we obtain in $\mathbb{Q}$ on the one hand

$$\lambda_n < r < s < \varrho_n \qquad (n \geqq 0);$$

by (1.3), (2.4), (1.4) we have

(4.9) $0 \leqq \alpha^{(2n+1)} - \alpha^{(2n)} \leqq 4^{-n}$, $0 \leqq \alpha^{(2n)} \leqq \alpha^{(1)}$ $\qquad (n \geqq 0)$

and similarly for β and γ; in $\mathbb{Q}$ this gives

$$\varrho_n - \lambda_n \leqq (\alpha^{(2n)} + 4^{-n})(\beta^{(2n)} + 4^{-n})(\gamma^{(2n)} + 4^{-n}) - \alpha^{(2n)}\beta^{(2n)}\gamma^{(2n)}$$
$$\leqq 4^{-n}(\alpha^{(1)}+1)(\beta^{(1)}+1)(\gamma^{(1)}+1) \qquad (n \geqq 0)$$

and for large n we have on the other hand

$$\varrho_n - \lambda_n < s - r;$$

this is a contradiction. Similarly the assumption "$>$" leads to a contradiction; finally (2.2) gives (4.8). We consider now the general case; by (4.4) and by the special case we have

$$|\alpha|\,|\beta\gamma| = |\alpha\beta|\,|\gamma|;$$

using this and also (4.2), (4.3), (4.4), (4.5) we settle the remaining 7 cases.

Theorem 4.2. For $\alpha \in K$, $\beta \in K$, $\gamma \in K$ we have

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma \quad \text{(distributivity)}. \tag{4.10}$$

Proof. We consider first the special case $\alpha > 0$, $\beta > 0$, $\gamma > 0$; we shall show that the assumption "$<$" as well as the assumption "$>$" gives a contradiction; then (2.2) gives (4.10).

"$<$". By Theorem 2.2 there exist $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ with

$$\alpha(\beta+\gamma) < r < s < \alpha\beta + \alpha\gamma;$$

by (1.3), (2.4), (2.5), (3.3), (4.6) and with

$$\lambda_n := \alpha^{(2n)}(\beta^{(2n)} + \gamma^{(2n)}),\ \varrho_n := \alpha^{(2n+1)}\beta^{(2n+1)} + \alpha^{(2n+1)}\gamma^{(2n+1)} \tag{4.11}$$

we obtain in $\mathbb{Q}$ at once

$$\lambda_n < r < s < \varrho_n \qquad (n \geqq 0); \tag{4.12}$$

by (4.9) this is a contradiction for large n.

"$>$". By Theorem 2.2 there exist $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ with

$$\alpha\beta + \alpha\gamma < r < s < \alpha(\beta+\gamma);$$

with (4.11) we obtain in $\mathbb{Q}$ again (4.12).

We consider now the general case. Trivially we may suppose $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. Since

$$(-\alpha)(\beta+\gamma) = -\alpha(\beta+\gamma) = \alpha((-\beta)+(-\gamma))$$
$$(-\alpha)\beta + (-\alpha)\gamma = -(\alpha\beta+\alpha\gamma) = \alpha(-\beta) + (\alpha(-\gamma)$$

by (4.5), (3.7) and since

$$\lambda = \mu \Leftrightarrow -\lambda = -\mu \quad (\lambda \in K, \mu \in K)$$

we have

$$\begin{aligned} \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma &\Leftrightarrow (-\alpha)(\beta+\gamma) = (-\alpha)\beta + (-\alpha)\gamma \\ &\Leftrightarrow \alpha((-\beta)+(-\gamma)) = \alpha(-\beta) + \alpha(-\gamma). \end{aligned} \tag{4.13}$$

In the general case we may by (4.13) and by $(-\beta)+(-\gamma) = -(\beta+\gamma)$ also suppose $\alpha > 0$, $\beta+\gamma > 0$. By (3.2) it is sufficient to prove (4.10) for

$$\alpha > 0,\ \beta > 0,\ \gamma < 0,\ \beta+\gamma > 0.$$

But then we have

$$\lambda := -\gamma > 0,\ \mu := \beta - \lambda = \beta + \gamma > 0$$

and (4.10) reads by (4.5) now

$$\alpha\mu = \alpha\beta - \alpha\lambda$$

or, by (3.5) and Theorem 3.2, equivalently

$$\alpha\mu + \alpha\lambda = \alpha(\mu + \lambda).$$

But this has been established in the special case.

Hence K is a commutative ring with respect to $+, \cdot$ without divisors of zero and has $\mathbb{Q}$ as subring.

The axiom of (Eudoxos and) Archimedes for K can immediately be verified by using Theorem 2.2, (4.6), (2.3).

§ 5. Division in K

For $\alpha \in K$, $\alpha > 0$ we have

$$0 < \alpha^{(2)} \leqq \alpha^{(2n+3)} \leqq \alpha^{(2n+1)} \leqq \alpha^{(1)} \text{ by (1.3), (2.4),}$$
$$0 < \tfrac{1}{\alpha^{(1)}} \leqq \tfrac{1}{\alpha^{(2n+1)}} \leqq \tfrac{1}{\alpha^{(2n+3)}} \leqq \tfrac{1}{\alpha^{(2)}} \qquad (n \geqq 0).$$

For $\alpha \in \mathbb{Q}$, $\alpha \neq 0$ we have

$$(5.1) \qquad \alpha^{-1} = \begin{cases} \sup\left\{\tfrac{1}{\alpha^{(2n+1)}} : n \geqq 0\right\} & \text{in case } \alpha > 0 \\ -|\alpha|^{-1} & \text{in case } \alpha < 0 \end{cases}$$

and $\alpha\alpha^{-1} = 1$. For $\alpha \in K \backslash \mathbb{Q}$ we use (5.1) as **Definition 5.1** of α^{-1}.

Theorem 5.1. For $\alpha \in K$, $\alpha \neq 0$ we have $\alpha\alpha^{-1} = 1$.

Proof. Let first $\alpha > 0$. By (5.1) we have

$$0 < \tfrac{1}{\alpha^{(2n+1)}} \leqq \alpha^{-1} \qquad (n \geqq 0).$$

By (1.3), (2.4) we have

$$0 < \tfrac{1}{\alpha^{(2j+1)}} \leqq \tfrac{1}{\alpha^{(2n+2)}} \qquad (j \geqq 0, n \geqq 0)$$

and by (5.1), (2.3) hence

$$0 < \alpha^{-1} \leqq \tfrac{1}{\alpha^{(2n+2)}} \qquad (n \geqq 0).$$

By (1.3), (2.4) we have

$$0 < \alpha^{(2n+2)} \leqq \alpha \leqq \alpha^{(2n+1)} \qquad (n \geqq 0).$$

Therefore (4.6) implies

$$0 < \tfrac{\alpha^{(2n+2)}}{\alpha^{(2n+1)}} \leqq \alpha\alpha^{-1} \leqq \tfrac{\alpha^{(2n+1)}}{\alpha^{(2n+2)}} \qquad (n \geqq 0).$$

By (1.3), (2.4), (1.4) we have

$$0 < 1 - \tfrac{\alpha^{(2n+2)}}{\alpha^{(2n+1)}} \leqq \tfrac{1}{4^n\alpha^{(2n+1)}} \leqq \tfrac{1}{4^n\alpha^{(2)}},$$
$$0 \leqq \tfrac{\alpha^{(2n+1)}}{\alpha^{(2n+2)}} - 1 \leqq \tfrac{1}{4^n\alpha^{(2n+2)}} \leqq \tfrac{1}{4^n\alpha^{(2)}} \qquad (n \geqq 0).$$

The assumption $1<\alpha\alpha^{-1} \vee \alpha\alpha^{-1}<1$ leads by Theorem 2.2, (2.3) in $\mathbb{Q}$ to the contradiction

$$\underset{r\in\mathbb{Q},}{\exists}\ \underset{n\in\mathbb{N}_0}{\forall}\ 1<r<1+\tfrac{1}{4^n\alpha^{(2)}}.$$

(2.2) gives the result. Let now $\alpha<0$. By (3.8) we have $|\alpha|=-\alpha>0$ and therefore $|\alpha|\,|\alpha|^{-1}=1$. By (5.1) and (4.5) we obtain

$$\alpha\alpha^{-1}=(-|\alpha|)(-|\alpha|^{-1})=|\alpha|\,|\alpha|^{-1}=1.$$

Hence K is a field with respect to $+,\cdot$ and has $\mathbb{Q}$ as subfield.

Final remarks

Altogether K is a complete ordered field and has $\mathbb{Q}$ as subfield. The elements $<a_0, a_1, \ldots>$ of K we call now real numbers and we write $\mathbb{R}$ instead of K.

At several occasions we have used a common principle: a theorem for $\mathbb{Q}$ is used to suggest a definition for $\mathbb{R}$. This was done in passing from $\mathbb{Q}=E'$ to $\mathbb{R}:=K$ and from Lemma 1.1, (3.1), (3.6), (4.1), (5.1) to the corresponding definition.

Let $\alpha=<a_0, a_1, \ldots>\in\mathbb{R}\setminus\mathbb{Q}$; by (2.4), (1.4) we have

$$|\alpha-\alpha^{(n)}|<|\alpha^{(n+1)}-\alpha^{(n)}|\leqq 2^{-n} \qquad (n\geqq 0)$$

and hence

$$\alpha=\lim_{n\to\infty}\alpha^{(n)};$$

by (1.1) this means

$$<a_0, a_1, a_2, \ldots> = a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}} := \lim_{n\to\infty}\left(a_0+\cfrac{1}{a_1+\ddots+\cfrac{1}{a_n}}\right)$$

(infinite continued fraction).

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