Opolka, Hans

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LIFTING OF CENTRAL 2-COCYCLES, CYCLOTOMIC SPLITTING FIELDS AND RATIONALITY OF GALOIS REPRESENTATIONS

Hans Opolka
TU Braunschweig
Institut für Analysis und Algebra
Pockelsstrasse 14
D - 38106 Braunschweig

e-mail address: h.opolka@tu-bs.de

Abstract: In this paper the problem of lifting a central 2-cocycle on a finite quotient group of the absolute Galois group of a field is related to the existence of a cyclotomic splitting field of a certain central simple algebra. Several examples are discussed and applications to the rationality problem for Galois representations are given.

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§1. Introduction

Let $\mathcal{G}$ be a profinite group. For any discrete $\mathcal{G}$-module M and any integer $q \geq 1$ let $H^q(\mathcal{G}, M)$ denote the q-th cohomology group of $\mathcal{G}$ with respect to M. We view $\mathbb{C}^*$ as a discrete $\mathcal{G}$-module with the trivial action of $\mathcal{G}$. Let G be a finite quotient group of $\mathcal{G}$ and let $(f) \in H^2(G, \mathbb{C}^*)$ be the class of a central 2-cocycle $f : G \times G \to \mathbb{C}^*$. The pair $(G, (f))$ is called a *central pair for* $\mathcal{G}$. Such a central pair is said to be *liftable* if (f) is contained in the kernel of the inflation homomorphism $\inf : H^2(G, \mathbb{C}^*) \to H^2(\mathcal{G}, \mathbb{C}^*)$.

(1.1) **Remarks** (a) By definition $H^2(\mathcal{G}, \mathbb{C}^*)$ is trivial if and only if every central pair for $\mathcal{G}$ is liftable

(b) For every finite abelian group G which is not cyclic the cohomology group $H^2(G, \mathbb{C}^*)$ is not trivial; see e.g. $[Y]$, Theorem 2.1, Corollary. Hence if such a group is realizable as a quotient group of $\mathcal{G}$ then there are central pairs $(G, (f))$ for $\mathcal{G}$ with nontrivial (f).

In $[O1]$ it was observed that the liftability of a central pair for a pro-p-Galois group is related to the existence of a cyclotomic splitting field for a certain central simple algebra. In the present paper we take up this idea in a more general form and apply it to the construction of fields of rationality of continuous finite dimensional representations of the absolute Galois group of certain fields. For other relations between central 2-cocycles on the absolute Galois group of a field k and the Brauer group of k see $[LO]$.

In the following we will make use of the profinite version of the Hochschild-Serre exact sequence: Let $\mathcal{U} \trianglelefteq \mathcal{G}$ be a closed normal subgroup of $\mathcal{G}$ and let M be a discrete $\mathcal{G}$-module. Then the following sequence is exact

$$(1.2)\ 1 \to H^1(\mathcal{G}/\mathcal{U}, M^{\mathcal{U}}) \stackrel{\inf}{\to} H^1(\mathcal{G}, M) \stackrel{res}{\to} H^1(\mathcal{U}, M)^{\mathcal{G}/\mathcal{U}} \stackrel{tr}{\to}$$

$$\stackrel{tr}{\to} H^2(\mathcal{G}/\mathcal{U}, M^{\mathcal{U}}) \stackrel{\inf}{\to} H^2(\mathcal{G}, M)$$

Here $N^{\mathcal{V}} := \{n \in N : s(n) = n \text{ for all } s \in \mathcal{V}\}$ for any profinite group $\mathcal{V}$ and any discrete $\mathcal{V}$-module N , inf is the corresponding inflation homomorphism, res is the corresponding restriction homomorphism and tr is the transgression homomorphism corresponding to the group extension $1 \to \mathcal{U} \to \mathcal{G} \to \mathcal{G}/\mathcal{U} \to 1$; for details comp. $[SH]$, II, §4, p. 51.

§2. Lifting and embedding problems

Denote by W the group of all roots of unity in $\mathbb{C}$ and for any positive integer d let $W_d \leq W$ denote the subgroup of all roots of unity whose orders divide d. Let $\mathcal{G}$ be a profinite group and let $(G,(f))$ be a central pair for $\mathcal{G}$. Denote by $m((f)) = m$ the order of (f). A central 2-cocycle $c : G \times G \to W_m$ is said to *belong to* (f) if (f) is the image of $(c) \in H^2(G, W_m)$ under the homomorphism $H^2(G, W_m) \to H^2(G, \mathbb{C}^*)$ which is induced by the inclusion $W_m \subset \mathbb{C}^*$. The existence of c was observed by I. Schur: There is a function $\alpha : G \to \mathbb{C}^*$ such that $f^m = \delta\alpha$. Choose a function $\beta : G \to \mathbb{C}^*$ such that $\beta^m = \alpha$ and put $c := f \cdot \delta(1/\beta)$. If $\widetilde{m}$ is a multiple of m denote by

$$c_{m,\widetilde{m}} : G \times G \stackrel{c}{\to} W_m \hookrightarrow W_{\widetilde{m}}$$

the central 2-cocycle which is obtained from c by composing c with the embedding $W_m \hookrightarrow W_{\widetilde{m}}$. The embedding problem $E(G,(f),c)$ for $\mathcal{G}$ which is defined by the group extension $G(c)$ corresponding to c is said to be *weakly solvable* if there is a multiple $\widetilde{m}$ of m such that the embedding problem $E(G,(f),c_{m,\widetilde{m}})$ for $\mathcal{G}$ which is defined by the group extension $G(c_{m,\widetilde{m}})$ corresponding to $c_{m,\widetilde{m}}$ is solvable, i.e. there is a homomorphism $\varphi : \mathcal{G} \to G(c_{m,\widetilde{m}})$ such that φ composed with the natural epimorphism $G(c_{m,\widetilde{m}}) \to G$ is equal to the given epimorphism $\mathcal{G} \to G$.

(2.1) **Proposition** *Let* $(G,(f))$ *be a central pair for* $\mathcal{G}$ *with nontrivial* (f). *Then the following statements are equivalent:*
(a) $(G,(f))$ *is liftable*

(b) There is a central 2-cocycle c belonging to (f) such that the central embedding problem $E(G,(f),c)$ for $\mathcal{G}$ is weakly solvable

If statement (b) holds then the smallest multiple $\widetilde{m}$ of m such that there is a central 2-cocycle c belonging to (f) for which the embedding problem $E(G,(f),c_{m,\widetilde{m}})$ for $\mathcal{G}$ is solvable is called the *lifting index of* $(G,(f))$; it is denoted by $l((f))$.

A proof of (2.1) could be given by following $[O1]$, §3, proof of (3.1). However, for the sake of completeness and for other reasons which will become clear later on we will give a slightly different proof which makes use of the Hochschild-Serre exact sequence and which is more in the spirit of the present paper; for the use of the Hochschild-Serre exact sequence in context of embedding problems comp. $[H]$, 2.1.

Proof of (2.1): Assume that statement (a) holds. Let c be a central cocycle belonging to (f). Then the assumption implies that there is a continuous function $\alpha : \mathcal{G} \to \mathbb{C}^*$ such that $\inf_G^{\mathcal{G}}(c) = \delta\alpha$. Since $c^m = 1$ the function $\alpha^m : \mathcal{G} \to \mathbb{C}^*$ is a continuous character, of order n say. Then $\widetilde{m} := mn$ has the property that the class of the central 2-cocycle $c_{m,\widetilde{m}}$ is contained in the kernel of the inflation homomorphism $\inf : H^2(G, W_{\widetilde{m}}) \to H^2(\mathcal{G}, W_{\widetilde{m}})$. Hence the Hochschild-Serre exact sequence (1.2) applied to the discrete $\mathcal{G}$-module $M = W_{\widetilde{m}}$ with the trivial group action and to $\mathcal{U} :=$ kernel of the given epimorphism $\mathcal{G} \to G$ yields a G-invariant character $\chi : \mathcal{U} \to W_{\widetilde{m}}$ such that $tr(\chi) = (c_{m,\widetilde{m}})$. Now it follows from the theory of group extensions that there is a solution $\varphi : \mathcal{G} \to G(c_{m,\widetilde{m}})$ of the embedding problem $E(G,(f),c_{m,\widetilde{m}})$ for $\mathcal{G}$ with the property that the restriction of φ to $\mathcal{U}$ equals χ, comp. $[AT]$, chapter 13, Theorem 2.

Conversely assume that statement (b) holds. Then there is a multiple $\widetilde{m}$ of m such that the embedding problem $E(G,(f),c_{m,\widetilde{m}})$ for $\mathcal{G}$ is solvable. If $\varphi : \mathcal{G} \to G(c_{m,\widetilde{m}})$ is a solution then again by $[AT]$, chapter 13, Theorem 2, the restriction of φ to $\mathcal{U}$ is a G-invariant character $\chi : \mathcal{U} \to W_{\widetilde{m}}$ such that $tr(\chi) = (c_{m,\widetilde{m}})$. Applying the Hochschild-Serre exact sequence (1.2) we see that $(c_{m,\widetilde{m}})$ is contained in the kernel of the inflation homomorphism $\inf : H^2(G, W_{\widetilde{m}}) \to H^2(\mathcal{G}, W_{\widetilde{m}})$. Hence $(G,(f))$ is liftable.

(2.2) **Remark** Assume that $(G,(f))$ is a central pair for $\mathcal{G}$ and let $c : G \times G \to W_m$ be a central 2-cocycle belonging to (f). If the central embedding problem $E(G,(f),c)$ for $\mathcal{G}$ is weakly solvable and if $c' : G \times G \to W_m$ is another central 2-cocycle belonging to (f) then the central embedding problem $E(G,(f),c')$ for $\mathcal{G}$ is weakly solvable too.

Proof : There is a function $\alpha : G \to \mathbb{C}^*$ such that $c' = \delta\alpha \cdot c$ and such that $\alpha^m : G \to \mathbb{C}^*$ is a homomorphism. By assumption there is a multiple $\widetilde{m}$ of m and a function $\beta : \mathcal{G} \to W_{\widetilde{m}}$ such that $\inf_G^{\mathcal{G}}(c_{m,\widetilde{m}}) = \delta\beta$. Let n denote any multiple of $\widetilde{m} \cdot \exp(G)$. Then $(\inf_G^{\mathcal{G}}(\alpha) \cdot \beta)^n = 1$ and therefore $(c'_{m,n}) \in H^2(G, W_n)$

is contained in the kernel of the inflation homomorphism $\inf_G^{\mathcal{G}} : H^2(G, W_n) \to H^2(\mathcal{G}, W_n)$. Hence the embedding problem $E(G,(f),c'_{m,n})$ for $\mathcal{G}$ is solvable. This shows that the embedding problem $E(G,(f),c')$ for $\mathcal{G}$ is weakly solvable.

§3. Lifting problems for profinite Galois groups and central simple algebras

Now we assume that $\mathcal{G} = \mathcal{G}_k$ is the absolute Galois group of a field k of characteristic 0, i.e. $\mathcal{G} = \mathcal{G}_k = G(\overline{k}/k)$ is the Galois group of an algebraic closure $\overline{k}$ of k. For every subextension K/k of $\overline{k}/k$ let $\mathcal{G}_K = G(\overline{k}/K)$ denote the Galois group of the extension $\overline{k}/K$. We identify W with the group of all roots of unity $\mu_{\overline{k}}$ of $\overline{k}$, and thereby for any positive integer d we get an identification of W_d with the group of all roots of unity μ_d of order dividing d in $\overline{k}$. Let G be a finite quotient group of $\mathcal{G}_k$. So $G = G(K/k)$ is the Galois group of a finite Galois subextension K/k of $\overline{k}/k$. Let $c : G \times G \to W_d$ be a central 2-cocycle belonging to (f). Put $G_d := G(K(\mu_d)/k(\mu_d))$ and let

$$A(c) := (K(\mu_d)/k(\mu_d), c_d)$$

denote the crossed product algebra corresponding to the restriction $c_d : G_d \times G_d \to \mu_d$ of c to G_d. $A(c)$ is a central simple $k(\mu_d)$-algebra. In cohomological terms the class of the algebra $A(c)$ in the Brauer group $Br(k(\mu_d))$ is the image of $(c) \in H^2(G, \mu_d)$ under the following sequence of homomorphisms

$$(3.1) \quad \alpha_d : H^2(G, \mu_d) \overset{\inf}{\to} H^2(\mathcal{G}_k, \mu_d) \overset{res}{\to} H^2(\mathcal{G}_{k(\mu_d)}, \mu_d) \hookrightarrow Br(k(\mu_d))$$

where the last homomorphism is given by the crossed product construction; comp. $[H]$, p. 88.

Assume that $(f) \in H^2(G, \mathbb{C}^*)$ is nontrivial of order m and let $c : G \times G \to \mu_m$ denote a central 2-cocycle belonging to (f). Put $A((f), c) := A(c)$. A finite subextension $E/k(\mu_m)$ of $\overline{k}/k(\mu_m)$ is called a *splitting field for* $(G,(f))$ if there is a central 2-cocycle c belonging to (f) such that E is a splitting field for the algebra $A(c)$.

(3.2) **Proposition** *Assume that $(G,(f))$ is a central pair for $\mathcal{G}_k$ with nontrivial (f)* of order m. *Then the following statements hold.*

(a) If $(G,(f))$ is liftable then there is a multiple $\widetilde{m}$ of m such that the cyclotomic field $k(\mu_{\widetilde{m}})$ is a splitting field for $(G,(f))$

(b) If there is a multiple $\widetilde{m}$ of m such that $k(\mu_{\widetilde{m}})$ is a splitting field for $(G,(f))$ and if the extension $k(\mu_{2^t})/k$, where 2^t is the maximal power of 2 dividing $\widetilde{m}$, is cyclic, then $(G,(f))$ is liftable with lifting index dividing $\widetilde{m}$

Proof: (a) According to (2.1) there is a central 2-cocycle c belonging to (f) and a multiple $\widetilde{m}$ of m such that the the embedding problem $E(G,(f),c_{m,\widetilde{m}})$ for

$\mathcal{G}_k$ is solvable. Applying the Hochschild-Serre exact sequence (1.2) or $[H]$, 1.1, this implies that $(c_{m,\widetilde{m}}) \in H^2(G, \mu_{\widetilde{m}})$ is contained in the kernel of the inflation homomorphism $\inf : H^2(G, \mu_{\widetilde{m}}) \to H^2(\mathcal{G}_k, \mu_{\widetilde{m}})$. Using the homomorphism $\alpha_{\widetilde{m}}$ defined under (3.1) we have

$$\alpha_{\widetilde{m}}((c_{m,\widetilde{m}})) = (A((f), c) \otimes_{k(\mu_m)} k(\mu_{\widetilde{m}})) = 1;$$

comp. also $[H]$, 3.2. Hence $k(\mu_{\widetilde{m}})$ is a splitting field for $(G, (f))$.

(b) Since we are dealing with *central* 2-cocycles we may and do assume that $\widetilde{m}$ is a prime power; comp. $[H]$, 1.2. According to $[H]$, 3.8, the assumption that the extension $k(\mu_{2^t})/k$ is cyclic assures that the solvability of the embedding problem $E(G, (f), c_{m,\widetilde{m}})$ and therefore the liftability of $(G, (f))$ follows from the fact that $k(\mu_{\widetilde{m}})$ is a splitting field for $(G, (f))$.

(3.3) **Corollary** *Assume that the field k contains the group $\mu_{\overline{k}}$ of all roots of unity in $\overline{k}$. Then a central pair $(G, (f))$ for $\mathcal{G}_k$ is liftable if and only if k is a splitting field for $(G, (f))$.*

Remark This result follows also from the fact that under the assumption $\mu_{\overline{k}} \le k^*$ the group $H^2(\mathcal{G}_k, \mathbb{C}^*) \cong H^2(\mathcal{G}_k, \mu_{\overline{k}})$ is isomorphic to the Brauer group of k.

(3.4) **Examples** In the following examples we make use of various results on the inverse problem of Galois theory. For an account of these results see $[K]$, Chapter 3, §3, and the references mentioned there.

(a) Assume that k is an algebraic function field of transcendence degree 1 over an algebraically closed field k_0 of characteristic 0. Then by well known results of Tsen $[TS1]$, $[TS2]$ the field k has the property (C1) and therefore the Brauer group of k is trivial; comp. also $[SE4]$, Chapter II, §3. Hence according to (3.2), (b), every central pair $(G, (f))$ for $\mathcal{G}_k$ is liftable. We recall the well known fact that in the special case $k = \mathbb{C}(t)$ every finite group G is realizable as a quotient group of $\mathcal{G}_k$; this follows from the result that $\mathcal{G}_k$ is free on continuum many generators, comp. $[SP]$, Chapter 10.9. Hence for $k = \mathbb{C}(t)$ there are many central pairs $(G, (f))$ for $\mathcal{G}_k$ with nontrivial (f).

(b) Assume that $k = \mathbb{R}(t)$ is the rational function field in one variable over $\mathbb{R}$. It follows from the forementioned results of Tsen that every central simple k-algebra is split by $\mathbb{C} = \mathbb{R}(\sqrt[2]{-1})$. Hence according to (3.2),(b) every central pair $(G, (f))$ for $\mathcal{G}_k$ is liftable. We note also that every finite group G is realizable as a quotient group of $\mathcal{G}_k$, comp. $[SE2]$, p. 92. Hence there are many central pairs $(G, (f))$ for $\mathcal{G}_k$ with nontrivial (f).

(c) If k is obtained from a number field k_0 by adjoining to k_0 all roots of unity $\mu_{\overline{k_0}}$ then the Brauer group of k is trivial, comp. e.g. $[D]$, Chapter VII, §5, Satz 4, and therefore according to (2.4) every central pair for $\mathcal{G}_k$ is liftable; this follows also from $[IW]$. Moreover there are finite groups G with nontrivial

$H^2(G, \mathbb{C}^*)$ which are realizable as quotient groups of $\mathcal{G}_k$, comp. $[IW]$. Hence there are central pairs $(G, (f))$ for $\mathcal{G}_k$ with nontrivial (f).

(d) If k is a local or global number field then $H^2(\mathcal{G}_k, \mathbb{C}^*) = \{1\}$, comp. $[SE1]$, §6. Therefore every central pair for $\mathcal{G}_k$ is liftable. Moreover there are central pairs $(G, (f))$ for $\mathcal{G}_k$ with nontrivial (f) because there are finite groups G with nontrivial $H^2(G, \mathbb{C}^*)$ which are realizable as quotient groups of $\mathcal{G}_k$; see $[K]$, 3.3, 3.4, 3.6, and the literature mentioned there.

§4. Lifting and rationality of Galois representations

Let $\mathcal{G}$ be a profinite group. By a linear resp. projective representation of $\mathcal{G}$ over a subfield E of $\mathbb{C}$ we mean a continuous homomorphism $D : \mathcal{G} \to GL(n, E)$ resp. $P : \mathcal{G} \to PGL(n, E)$ for some positive integer n which is called the degree of D resp. P, where the topology on $\mathcal{G}$ is of course the profinite topology and the topology on $GL(n, E)$ resp. $PGL(n, E)$ is the discrete topology. The kernel of a linear resp. projective representation of $\mathcal{G}$ is a closed normal subgroup of $\mathcal{G}$ which is of finite index in $\mathcal{G}$. Hence every linear resp. projective representation of $\mathcal{G}$ can be viewed as a linear resp. projective representation of a finite quotient group of $\mathcal{G}$, and therefore all concepts which are familiar from the representation theory of finite groups, e.g. irreducibility, rationality, can be used for linear resp. projective representations of $\mathcal{G}$.

In $[O2]$ the problem of rationality for representations $D : \mathcal{G} \to GL(n, \mathbb{C})$ of certain profinite groups $\mathcal{G}$ was related to the torsion part of their maximal profinite abelian quotient groups $\mathcal{G}^{ab}$. In this way crude information about the fields of rationality could be obtained. The approach below is different. It makes use of the connection between lifting and central simple algebras but is restricted to representations of certain profinite Galois groups.

Assume that $P : \mathcal{G} \to PGL(n, E)$ is a projective representation. Put $G(P) := \mathcal{G}/Ker(P)$. Then P determines the class of a 1-cocycle

$$(P) \in H^1(G(P), PGL(n, E)),$$

and the coboundary map

$$\delta : H^1(G(P), PGL(n, E)) \to H^2(G(P), E^*),$$

comp. $[SE3]$, p. 124/125, determines the class of a central 2-cocycle $(f_P) := \delta((P)) \in H^2(G(P), E^*)$. For $E = \mathbb{C}$ we get a central pair $(G(P), (f_P))$ for $\mathcal{G}$ which is liftable if and only if there is a linear representation $D : \mathcal{G} \to GL(n, \mathbb{C})$ such that the corresponding projective representation

$$\overline{D} : \mathcal{G} \xrightarrow{D} GL(n, \mathbb{C}) \xrightarrow{\pi} PGL(n, \mathbb{C}),$$

where π denotes the natural projection, is ismorphic to P. For the proof of the following proposition compare also [J], Lemma 1.2, p. 9, and [$O2$], Lemma 9.

(4.1) **Proposition** *Assume that* $E \subset \mathbb{C}$ *is a subfield and that* $P : \mathcal{G} \to PGL(n, E)$ *is a projective representation. Then there is a field extension* E'/E *of degree dividing* n^r*, where* $r = r(P)$ *is the rank of the factor commutator group* $G(P)/G(P)'$*, such that the image of* $(f_P) \in H^2(G(P), E^*)$ *under the homomorphism* $H^2(G(P), E^*) \to H^2(G(P), E'^*)$ *which is induced by the embedding* $E \hookrightarrow E'$ *contains a central 2-cocycle* $f^{'}$ *with the property that all values of* f' *are roots of unity of order dividing* n*. Moreover, if the central pair* $(G(P), (f'))$ *for* $\mathcal{G}$ *is liftable with lifting index* $l = l((f^{'}))$ *and if* P *is absolutely irreducible then there is an absolutely irreducible linear representation* $D : \mathcal{G} \to GL(n, E'(W_l))$ *such that* $\overline{D}$ *is isomorphic to* P *over* $E'(W_l)$*; and for every linear representation* D *of* $\mathcal{G}$ *with the property that* $\overline{D}$ *is isomorphic to* P *over* $\mathbb{C}$ *the field which is obtained from* $\mathbb{Q}$ *by adjoining to* $\mathbb{Q}$ *all values of the character of* D *contains the field* $\mathbb{Q}(W_l)$.

Proof: Let $T : G(P) \to GL(n, E)$ be a mapping such that $T(s)T(t) = f_P(s,t)T(st)$ for all $s, t \in G(P)$. For every $s \in G(P)$ choose an n-th root of $\det(T(s))$ and call it $\beta(s)$. Define $f' := \delta\beta^{-1} \cdot f_P$ and $T' := \beta^{-1} \cdot T$. Then all values of f' are n-th roots of unity, and for every $s \in G(P)$ all matrix coefficients of $T'(s)$ belong to the field E' which is obtained from E by adjoining to E all $\beta(s_1), ..., \beta(s_r)$, where $s_1, ..., s_r \in G(P)$ are such that

$$s_1 \bmod G(P)^{'}, ..., s_r \bmod G(P)^{'} \in G(P)/G(P)'$$

generate the quotient group $G(P)/G(P)'$. Moreover the projective representation $\overline{T'} : G(P) \xrightarrow{T} GL(n, E') \to PGL(n, E')$ is isomorphic over E' to the projective representation P. The lifting assumption implies that there is a $G(P)$-invariant character $\chi : Ker(P) \to W_l$ such that $tr(\chi) = (f')$. Clifford's theory shows that there is an absolutely irreducible linear representation $D : \mathcal{G} \to GL(n, E'(W_l))$ such that $\mathrm{Re}\, s^{\mathcal{G}}_{Ker(P)}(D) \cong n.\chi$ and such that $\overline{D}$ is isomorphic to P over $E'(W_l)$; comp. [CL]. Moreover, since $l = l((f'))$ divides the order of every $G(P)$-invariant character $\chi : Ker(P) \to \mathbb{C}^*$ with the property $tr(\chi) = (f')$ the last assertion of the proposition follows.

(4.2) **Corollary** *For every irreducible projective representation* $P : \mathcal{G} \to PGL(n, \mathbb{C})$ *such that the central pair* $(G(P), (f_P))$ *for* $\mathcal{G}$ *is liftable there is a multiple* g *of* $|G(P)|$ *and a linear representation* $D : \mathcal{G} \to GL(n, \mathbb{Q}(W_g))$ *such that* $\overline{D} \cong P$.

Proof: According to $[R]$ there is a projective representation $P' : G(P) \to GL(n, \mathbb{Q}(W_{|G(P)|}))$ such that $P' \cong P$ and such that all values of $f_{P'}$ are roots of unity of order dividing $|G(P)|$. The assertion follows from (3.2),(b).

Two linear representations D, D' of $\mathcal{G}$ over the subfield $E \subset \mathbb{C}$ are said to belong to the same *genus* if there is a onedimensional representation $\lambda : G \to E^*$ such that D' is isomorphic to $\lambda \otimes D$. This is an equivalence relation, and the corresponding equivalence classes are called genera.

Obviously, if D, D' belong to the same genus then $\overline{D} \cong \overline{D'}$ and therefore $(f_{\overline{D}}) = (f_{\overline{D'}})$. If P is a projective representation of $\mathcal{G}$ over $\mathbb{C}$ and if the central pair $(G(P), (f_P))$ for $\mathcal{G}$ is liftable then any two linear representations D, D' of $\mathcal{G}$ over $\mathbb{C}$ with the property $\overline{D} \cong P \cong \overline{D'}$ belong to the same genus.

(4.3) **Examples** (a) For the field $k = \mathbb{C}(t)$ the genus of every irreducible linear representation D of $\mathcal{G}_k$ over $\mathbb{C}$ contains a representation which is ratonal over $\mathbb{Q}(W_{|G|})$. In order to see this we recall, see example (3.4),(a), that the Brauer group of k is trivial and that therefore the lifting index of the central pair $(G(\overline{D}), (f_{\overline{D}}))$ is equal to the order of $(f_{\overline{D}}) \in H^2(G(\overline{D}), \mathbb{C}^*)$. According to $[R]$ there is a projective representation $\mathcal{G}_k \to PGL(n, \mathbb{Q}(W_{|G(\overline{D})|}))$ which is isomorphic to $\overline{D}$ over $\mathbb{C}$. Since the order of $(f_{\overline{D}})$ divides the order of $G(\overline{D})$ the assertion follows.

(b) For the field $k = \mathbb{R}(t)$ the genus of every irreducible linear representation D of $\mathcal{G}_k$ over $\mathbb{C}$ contains a representation which is rational over $\mathbb{Q}(W_{\mathrm{lcm}(4,|G(\overline{D})|)})$. In order to see this we recall, see example (3.4),(b), that every element in the Brauer group of k is split by $k(W_4)$ and that therefore the lifting index of the central pair $(G(\overline{D}), (f_{\overline{D}}))$ divides the lcm of 4 and the order of $(f_{\overline{D}}) \in H^2(G(\overline{D}), \mathbb{C}^*)$. Using again $[R]$ the assertion follows similarly as in example (a).

(c) If k is a field which is obtained from a number field k_0 by adjoining to k_0 all roots of unity in $\overline{k_0}$ then a similar argument as in example (a) shows that the genus of every irreducible linear representation D of $\mathcal{G}_k$ over $\mathbb{C}$ contains a representation which is rational over $\mathbb{Q}(W_{|G(\overline{D})|})$.

(d) If k is a number field then the genus of every irreducible linear representation D of $\mathcal{G}_k$ over $\mathbb{C}$ contains a representation which is rational over $\mathbb{Q}(W_g)$, where g is a multiple of $|G(\overline{D})|$ with the property that the field $k(\mu_g)$ is a splitting field for $(G(\overline{D}), (f_{\overline{D}}))$. Because if g is chosen in this way it can be shown, by making use of the global duality theorem of Tate and Poitou $[P]$, that the lifting index of $(G(\overline{D}), (f_{\overline{D}}))$ divides $2g$, and the reasoning from example (a) yields the assertion. The existence of g such that $k(\mu_g)$ is a splitting field for $(G(\overline{D}), (f_{\overline{D}}))$ follows e.g. from $[T]$, p. 192, proof of lemma. Hence in the case of number fields the multiple of the order of $G(\overline{D})$ which occurs in (4.2) can be made rather explicit by using the theory of central simole algebras over number fields.

(e) It follows from e.g. [$SE5$], section 1, that there is an absolutely irreducible projective representation of $P : \mathcal{G}_{\mathbb{Q}} \to PGL(2, \mathbb{Q}(\sqrt[2]{5}, \sqrt[2]{-1}))$ such that the image of P is isomorpic to the alternating group A_5. The lifting index of $(G(P), (f_P))$ divides 2^{l+1} where l is any positive integer such that $k(\mu_{2^l})$ is a splitting field for $(G(P), (f_P))$. Hence there is an absolutely irreducible linear representation $D : \mathcal{G}_{\mathbb{Q}} \to GL(2, \mathbb{Q}(\sqrt[2]{5}, \mu_{2^{l+1}}))$ such that $\overline{D} \cong P$.

Similar applications of central simple algebras have been made by the author in various different contexts, e.g. in [$O1$].

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