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Central simple algebras and Galois representations

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Abstract: This is a survey about connections between central simple algebras and Galois representations in the case of number fields

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§1. Regular crossed products and Galois representations

The connection between the theory of central simple algebras and Galois representations is based on a result of R. Brauer [B] which says that every such algebra is similar to a crossed product with a Galois 2-cocycle all of whose values are roots of unity. At first we describe the basic construction which leads to this connection, for details see also [O3].

Let \( k \) be a field, let \( \overline{k} \) be a separable algebraic closure of \( k \), and for every subextension \( k'/k \) of \( k/k \) let \( G_{k'} = G(k'/k) \) denote the profinite Galois group of the extension \( k'/k \). Denote by \( \mu_k \) the group of roots of unity in \( k \) and for every positive integer \( m \) let \( \mu_m \) denote the group of \( m \)-th roots of unity in \( \overline{k} \). It is well known that every associative finite dimensional central simple \( k \)-algebra \( A \) (c.s. \( k \)-algebra for short) such that the characteristic of \( k \) does not divide the exponent of \( A \) is similar to a crossed product \((K/k, c)\), where \( K/k \) is a finite Galois subextension of \( \overline{k}/k \) - with Galois group \( G = G(K/k) \) say - and where \( c : G = G(K/k) \rightarrow \mu_K \) is a Galois 2-cocycle, see [B], §6, Satz 10. Adopting a terminology from [B], §2, such a crossed product is called regular. It is also shown in [B], §6, that \( \mathfrak{A} \) is similar to a regular crossed product \((K/k, c)\) where the order of \( c \) divides the exponent of \( \mathfrak{A} \). Assume now that the characteristic of \( k \) is 0 and that \((K/k, c)\) is a regular crossed product. Denote by \( m = m(c) \) the order of \( c \), i.e. the smallest positive integer \( j \) such that \( c(\sigma, \tau)^j = 1 \) for all \( \sigma, \tau \in G = G(K/k) \). We are going to construct a set \( \mathfrak{R}(K/k, c) \) of isomorphism classes of irreducible continuous representations \( R : G_k \rightarrow GL(n, \overline{k}) \) - where \( G_k \) is regarded as a topological group with respect to the profinite topology and \( GL(n, \overline{k}) \) as a topological group with respect to the discrete topology - under the following assumption.

(1.1) Assumption \( H^2(G_{k'}, \mathbb{Q}/\mathbb{Z}) = \{0\} \) for every finite abelian subextension \( k'/k \) of \( \overline{k}/k \).
It is well known that this assumption holds if $k$ is a local or global number field, see [T] and [SE1], § 6; and - of course - it holds if $k$ is a field of cohomological dimension $\leq 1$, see [SE4], II, §3, especially 3.3 for examples. In order to construct $\mathcal{R}(K/k, c)$ we put $G_m := G(K/k(\mu_m))$, $c_m :=$ restriction of $c$ to $G_m \times G_m$; hence $c_m : G_m \times G_m \rightarrow k(\mu_m)^*$ is a central 2-co-cycle.

Let $T : G_m \rightarrow GL(n, k)$ be an irreducible continuous $c_m$-representation of $G_m$, compl. $[M]$; so we have $T(\sigma)T(\tau) = c_m(\sigma, \tau)T(\sigma\tau)$ for all $\sigma, \tau \in G_m$. It follows from (1.1) that $T$ has a lifting, i.e. there is a continuous irreducible linear representation $D : G_{k(\mu_m)} \rightarrow GL(n, k)$ such that the corresponding projective representations $\overline{D}$ and $\overline{T}$ coincide, see e.g. [SE1], §6. Denote by $\mathcal{R}_D$ the set of all isomorphism classes $(R)$ of irreducible continuous linear representations $R$ of $G_k$ of finite degree such that the restriction of $R$ to $G_{k(\mu_m)}$ contains $D$ as an irreducible constituent, and let $\mathcal{R}(K/k, c)$ denote the union of all sets $\mathcal{R}_D$ where $D$ is any linear representation of $G_{k(\mu_m)}$ of finite degree which lifts an irreducible $c_m$-representation of $G_m$. Using the Clifford Mackey theory, see [CL] and [M] or the corresponding sections in [CR], one proves the following proposition, for details see [O3], §1.

(1.2) Proposition The degree of every $(R) \in \mathcal{R}(K/k, c)$ divides the degree $(K : k)$.

§2. Regular crossed products and Galois representations in the case of number fields

Let $k$ be a number field. A continuous linear or projective representation $D$ of $G_k$ over $k$ of finite degree is said to be unramified outside a given finite set of places $S$ of $k$, if for all places of $k$ which do not belong to $S$ the corresponding inertia subgroups are contained in the kernel of $D$. As is well known, using e.g. the nonabelian version of the "Führerdiskriminantenproduktformel" in [S2], VI, §3, Cor. 2, p. 104, the following result is an easy consequence of a result of I. Schur [S] and a variant of the well known result of Hermite and Minkowski according to which there are only finitely many number fields with a given discriminant, see e.g. [K], Satz 2.13.6, S. 57.

(2.1) Proposition Let $k$ be a number field, let $S$ be a finite set of places of $k$, let $n$ be a positive integer and let $E/k$ be a subextension of $k$ of finite degree. Then there are only finitely many isomorphism classes of continuous linear representations $R : G_k \rightarrow GL(n, k)$ such that $R$ is unramified outside $S$ and such that all values of the character of $R$ belong to $E$.

In view of this result it seems worthwhile to investigate rationality and ramification properties of representations of the form constructed above from regular crossed products. For any c.s. $k$-algebra $\mathfrak{A}$ denote by $S_\mathfrak{A}$ the finite set of all places of $k$ at which $\mathfrak{A}$ does not split. Let $(K/k, c)$ be a regular crossed product which is similar to $\mathfrak{A}$. Denote by $S_{K/k}$ the finite set of all places which are ramified in $K/k$. Since all values of $c$ are roots of unity it follows from the local theory
of central simple algebras, see e.g. [D], VII, §2, especially Satz 3, p.112, that \( \mathfrak{A} \) splits at all places which are unramified in \( K/k \). So we have \( S_{\mathfrak{A}} \subset S_{K/k} \). It was observed by Hasse [H], see also [D], VII, Satz 4, S. 118, that there is a smallest multiple \( g = g(\mathfrak{A}) \) of the exponent \( \exp(\mathfrak{A}) \) of \( \mathfrak{A} \) such that \( k(\mu_r) \) is a splitting field of \( \mathfrak{A} \); namely, by the local theory of c.s. algebras and by the local-global principle for c.s. algebras, see [D], VII, §5, Satz 1, p. 117, \( g \) is the smallest positive multiple of the exponent of \( \mathfrak{A} \) such that the local degrees \( (k_v(\mu_r) : k_v) \) are divisible by \( \exp(\mathfrak{A}) \) for all \( v \in S_{\mathfrak{A}} \). Let \( m = m(c) \) denote the order of \( c \). Define the cyclotomic index \( \tilde{g} := \tilde{g}(K/k, c) \) of \( (K/k, c) \) by \( \text{l.c.m.}(m(c), g(\mathfrak{A})) \) if \( m \) is odd and by \( \text{l.c.m.}(4, m(c), g(\mathfrak{A})) \) if \( m \) is even. Using the profinite version of the exact Hochschild-Serre sequence, see e.g. [SH], chapter II, §4, and results in [P], section 2, one proves

\[(2.2) \quad \text{Proposition} \quad \text{There is } (R) \in \mathfrak{A}(K/k, c) \text{ such that all values of the character of } R \text{ belong to } k(\mu_{\exp(G(K/k))\tilde{g}(K/k, c)}) .\]

For every positive integer \( t \) denote by \( S_t \) the set of all places of \( k \) which divide \( t \) and the infinite place of \( \mathcal{Q} \), and for every finite set of places \( S \) of \( k \) let \( k_S/k \) denote the maximal Galois subextension of \( K/k \) which is unramified outside \( S \).

\[(2.3) \quad \text{Assumption} \quad \text{Let } q \text{ be a prime number. Then for every finite set of places } S \text{ of } k \text{ which contains } S_q \text{ the following statement holds:} \]

\[L(S, q) : H^2(G(k_S/k'), Q_q/Z_q) = \{0\} \text{ for every finite abelian subextension } k'/k \text{ of } k_S/k.\]

It is known that \( L(S, q) \), which is related to the Leopoldt-conjecture, is true if \( k \) is an abelian extension of \( \mathcal{Q} \); see [BR] in connection with [MK].

Let \( m = q^i \) be a power of a prime number \( q \) and let \( S \) be a finite set of places of \( k \) containing \( S_m \cup S_{K/k} \). Then under assumption (2.3) there is a smallest positive integer \( \lambda = \lambda(K/k, c) \) such that the central embedding problem for \( G(k_S/k(\mu_m)) \) which is defined by the cocycle class \( (c_m) \in H^2(G_m, \mu_m) \) is weakly solvable with respect to \( \lambda \), i.e. the central embedding problem for \( G(k_S/k(\mu_m)) \) corresponding to the image of \( (c_m) \) under the homomorphism of cohomology groups with respect to the trivial group action \( H^2(G_m, \mu_q) \rightarrow H^2(G_m, \mu_{q^i} \lambda) \) which is induced by the embedding \( \mu_{q^i} \rightarrow \mu_{q^i + \lambda} \) is solvable; \( l := l(K/k, c) := q^i \lambda \) is called the \textit{Leopoldt-index} of the regular crossed product \( (K/k, c) \). A more detailed investigation of closely related invariants is contained in [NO]. For any continuous representation \( R \) of \( G_k \) denote by \( S_R \) the set of all places of \( k \) which are ramified in the fixed field of the kernel of \( R \).

\[(2.4) \quad \text{Proposition} \quad \text{Under assumption (2.3) there is } (R) \in \mathfrak{A}(K/k, c) \text{ such that } S_R \subset S_{K/k} \text{ and such that all values of the character of } R \text{ belong to } k(\mu_{\exp(G)}l(K/k, c)).\]
§3. Finite symplectic Galois modules and regular crossed products

Let $k$ be a field of characteristic 0 such that assumption (1.1) holds. Let $A$ be a finite continuous $G_k$-module which is equipped with a nondegenerate symplectic $G_k$-equivariant pairing $\omega : A \times A \to \mu_m$. It can be shown that there is a central 2-cocycle $f : A \times A \to \mu_m$ such that $\omega(x, y) = f(x, y)/f(y, x)$, $x, y \in A$, and that every cocycle class $(\alpha) \in H^1(G_k, A)$ defines a unique element in the Brauer group $Br(k)$ of $k$ which can be represented by a regular crossed product $(K_{(\alpha)}/k, c)$, where $K_{(\alpha)}$ is a finite Galois splitting field of $(\alpha)$ and $c = c_{(\alpha), f} : G_{(\alpha)} \times G_{(\alpha)} \to \mu_m \subset K_{(\alpha)}^*$ is a Galois 2-cocycle on $G_{(\alpha)} := G(K_{(\alpha)}/k)$ all of whose values belong to $\mu_m$; see [O3], §3. Put $N_{(\alpha), f} := N(G_{(\alpha)}/k, c)$. Denote by $K_A$ the fixed field of the kernel of the action of $G_k$ on $A$.

We assume

$$\rho := res_{K_A}^G(\alpha)) \in Hom(G_{(\alpha)}, A)^{G(K_A/k)}$$

is surjective.

It is easily seen that there is $\alpha \in H^1(G_k, A)$ satisfying (3.1) provided $H^2(G(K_A/k), A) = 0$. In fact, according to [IK] there is a surjective solution $\phi$ of the embedding problem for $G_k$ which is defined by the semidirect product of the $G(K_A/k)$-module $A$ with $G(K_A/k)$. The restriction of $\phi$ to $G_{K_A}$ yields a surjective $\rho \in Hom(G_{K_A}, A)^{G(K_A/k)}$. Since $H^2(G(K_A/k), A) = \{1\}$ by assumption, the exact Hochschild-Serre sequence shows that there is $\alpha \in H^1(G_k, A)$ such that $\rho = res_{K_A}^G(\alpha))$. Examples of symplectic $G_k$-modules $A$ with trivial $H^2(G(K_A/k), A)$ arise naturally in the theory of elliptic curves; see [SE3]. Representations similar to those in $N_{(\alpha), f}$ have been constructed by a different method, which is implicit e.g. in [W1], in [O1]. For the algebraic framework see also [Z].

§4. Examples

In this section we describe various examples.

(1) Central pairs and Galois representations, see also [O2], [O3], [O4].

Let $k$ be a field of characteristic 0. Let $A$ be a finite abelian group of prime exponent and let $f : A \times A \to k^*$ be a central 2-cocycle. $(A, f)$ is called a central pair. As can be seen from [A] chapter V, section 4, p. 186ff, central pairs arise naturally in the theory of quadratic forms. We assume that $(A, f)$ has the following properties:

(a) The symplectic pairing $\omega_f : A \times A \to k^*$, $\omega_f(x, y) := f(x, y)/f(y, x)$, $x, y \in A$, which was introduced in [JM], §1, p.132, is nondegenerate; so especially all values of $\omega_f$ belong to $\mu_m$ where $m$ is the exponent of $A$, and $\mu_m \subset k^*$.

(b) The central pair $(A, f)$ is full, which means that the following conditions (i) and (ii) hold:

(i) There is a map $\alpha_f : A \to k^*$ such that

\[ \alpha_f(x)^{\text{ord}(x)} = \prod_{j=1}^{\text{ord}(x)} f(x, x^j) \text{ for all } x \in A \]

(ii) The degree of every \( \alpha_f(x) \), \( x \in A \), over \( k \) is the order \( \text{ord}(x) \) of \( x \), and the degree of the extension \( k_f/k \) which is generated over \( k \) by all \( \alpha_f(x) \), \( x \in A \), is the order of \( A \).

Then, if we consider \( A \) as a trivial \( G_k \)-module, the pair \((A, \omega_f)\) is a nondegenerate symplectic \( G_k \)-module. Moreover, the composition of maps

\[ \alpha : G_k \xrightarrow{\beta} \hat{A} \xrightarrow{\gamma} A, \]

where

\[ \beta(\sigma)(x) := \sigma(\alpha_f(x))/\alpha_f(x) \text{ for all } \sigma \in G_k, x \in A, \]

and

\[ \gamma(\lambda) := x_\lambda \text{ is such that } \lambda(y) = \omega_f(x_\lambda, y) \text{ for all } \lambda \in \hat{A}, y \in A, \]

defines a surjective homomorphism

\[ (\alpha) \in H^1(G_k, A) = \text{Hom}(G_k, A) \]

with the property \( k(\alpha) = k_f \). Let \( f_0 : A \times A \to \mu_m \) denote a central 2-cocycle such that \( \omega_f = \omega_{f_0} \). Then \( R_{(\alpha), f_0} \) is defined. This set has been constructed and investigated in [O2]. Especially the following results are shown there:

The character group \( \hat{G}_k \) acts transitively on \( R_{(\alpha), f_0} \). Every \((R) \in R_{(\alpha), f_0}\) has degree \(|A|^{1/2}\). If \( k \) is a number field there is \((R) \in R_{(\alpha), f_0}\) such that \( S(R) \subset \{v : v \text{ divides } m, v \text{ divides } a_f(x) \text{ for all } x \in A, v \text{ divides } \infty\} \), and all values of the character of \((R)\) belong to \( k(\mu_{\hat{g}}) \) where \( \hat{g} \) is the cyclotomic index of \((\alpha)\).

Moreover, as was observed in [O2], the results on automorphic induction in [AC] imply:

For a number field \( k \subset \mathbb{C} \) every \((R) \in R_{(\alpha), f_0}\) is cuspidal automorphic in the sense of [L]. More precisely the cuspidal automorphic representation corresponding to \((R) \in R_{(\alpha), f_0}\) is automorphically induced in the sense of [AC] by a continuous character \( \lambda \) of \( GL(1, \mathbb{A}_M) \) of finite order where \( M \subset k_f \) is the
fixed field corresponding to any maximal \( \omega \)-isotropic subgroup of \( A \) under the above isomorphism \( \alpha: G(k_f/k) \cong A \), and the character \( \lambda \) corresponds under the Artin map \( GL(1, \mathbb{A}_M) \to G_M^{ab} \) (see [AT]) to a continuous character \( \tilde{\lambda} \) of \( G_M \) such that the restriction of \( \tilde{\lambda} \) to \( G_{k_f} \) is the central character \( \gamma_R \) of \( (R) \), i.e. \( \gamma_R \) is the unique irreducible constituent of the restriction of \( R \) to \( G_{k_f} \).

(Here \( \mathbb{A}_K \) denotes the adele ring of the number field \( K \).)

In the special case \( A \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) the corresponding modular forms include those constructed by E. Hecke [HE] from indefinite binary quadratic forms, see [O2], section 4, and the literature mentioned there; and also certain wave forms, see [O4] and the literature mentioned there.

(2) Elliptic curves and Galois representations, see also [BF], [BU], [HA], [J], [O1], [O2], [O3].

Let \( X \) be an elliptic curve defined over \( k \). For any positive integer \( m \) denote by \( X_m \) the kernel of the multiplication by \( m \) homomorphism \( X(\mathbb{F}) \xrightarrow{m} X(\mathbb{F}) \).

The Weil-pairing \( \omega = \omega_m : X_m \times X_m \to \mu_m \) is a nondegenerate bilinear \( G_k \)-equivariant symplectic pairing, see [T]. Let \( f : A \times A \to \mu_m \) be a central 2-cocycle such that \( f(x, y)/f(y, x) = \omega(x, y) \), \( x, y \in A \). Every \( k \)-rational point \( P \in X(k) \setminus mX(k) \) defines an element \( \Delta(P) \in H^1(G_k, X_m) \), \( \Delta(P)(\sigma) = \sigma(Q) - Q \) for all \( \sigma \in G_k \), where \( Q \in X(\mathbb{F}) \) is such that \( mQ = P \). If the restriction \( \Delta(P)/G_{k(X_m)} \in \text{Hom}(G_k(X_m), X_m)^{G(k(X_m)/k)} \) is surjective, then, according to the above construction, the rational point \( P \) defines the set \( \mathcal{R}_{P,f} := \mathcal{R}_b(\Delta(P),f) \) of isomorphism classes of continuous irreducible representations of \( G_k \) over \( K \).

In the case of number fields Kummer theory for elliptic curves as developed in [BK] and [LG], chapter V, yields examples with surjective \( \Delta(P)/G_{k(X_m)} \). Similar representations have been constructed in a slightly different way in [O1].

They were investigated further in [J]. The construction of an odd 2-dimensional Galois representation of octahedral type of Artin-conductor 59\(^2\) in [HA] makes also use - at least implicitly - of an elliptic curve, namely \( X : y^2 = x^3 + 2x - 1 \). The construction and thorough investigation of odd 2-dimensional Galois representations of \( G_Q \) of octaedral type in [BF] and [BU] is based on elliptic curves \( X \) over \( Q \) and nontrivial elements in \( H^1(G_Q, X_2) \) which are interpreted in terms of 2-coverings of \( X \). For the theory of \( m \)-coverings of elliptic curves see [BS] and [C].

References


[AT] E. Artin, J. Tate: Class field theory, Benjamin, New York, 1967


[B] R. Brauer: Über die Konstruktion der Schießkörper, die von endlichem Rang in bezug auf ein gegebenes Zentrum sind, JRAM, 168, 1932, 44-64


[IM] N. Iwahori, H. Matsumoto: Several remarks on projective representations of finite groups, Journal of the Faculty of Science of the University of Tokyo, Sect. I, 10, 1964, 129-146


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